1415

THE JOURNAL

OF THE

INDIAN MATHEMATICAL SOCIETY

18 SEP 1930
MADRAS

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Volume XVIII, No. 10.

August 1930.

MADRAS:
PRINTED BY SRINIVASA VARADACHARI & CO.

Annual Subscription: Rs. 9.] [Single Copy: Rupee One and Annas Eight.

CONTENTS.

Pages

On Mr. Ramaswamy Aiyar's Topological Problem: A. Narasinga Rao—(*contd.*) ... ... 241—254

The Theory of Intermittent action and Band-Spectra in Infra-red: D. K. Sen and B. N. Biswas ... 255—262

A Theorem concerning Integral Functions of Order Unity and Type Positive: P. L. Srivastava ... 263—264

NOTES AND QUESTIONS:

New Light on Bhaskara's Chakravala or Cyclic Method of solving Indeterminate Equations of the Second Degree in two Variables: A. A. Krishnaswami Ayyangar, M.A., L.T. *225—248*

Solutions ... ... *249—255*

Questions for Solution *256*

A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. Contributors are requested to bear in mind that, at the present stage of mathematical research, hardly any paper is likely to be so completely original as to be independent of earlier work in the same direction; and that readers are often helped to appreciate the importance of a new investigation by seeing its connection with earlier results.

The principal results of a paper should, when possible, be enunciated separately and explicitly in the form of definite theorems.

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Lastly let us classify the several modes of connecting n villages simply, according to the *number of their terminii*. If $f(n, t)$ denote the number of these which have t specified villages and these only for terminii, the total number of methods of connection which have each just t terminii (unspecified) is ${}_nC_t \cdot f(n, t) = g(n, t)$ say. Allowing t to range from 2 to $n - 1$, which is the range of its possible values, we have:

$$\sum_{t=2}^{n-1} g(n, t) = F(n). \qquad \ldots \quad (3.4)$$

Other formulae analagous to (3.4) may be obtained by counting all modes of connection which have a particular village v_1 for a terminus (along with possibly other unspecified terminii). This number is $(n-1) \cdot F(n-1)$ as shown already. As there are altogether n village we have $n(n-1) \cdot F(n-1)$ modes of connection altogether. But in reckoning thus, each mode of connection gets counted as many times as it has terminii, so that we have:

$$\sum_{t=2}^{n-1} t \cdot g(n, t) = n(n-1) F(n-1). \qquad \ldots \quad (3.5)$$

And more generally, with r specified villages as terminii, (with possibly others also) there are $(n-r)^r \cdot F(n-r)$ modes of simply connecting n villages, since to get all such modes we have only to connect the remaining $(n-r)$ villages simply in all possible ways and connect each of the given r villages to any of the $n - r$. Taking all the ${}_nC_r$ combinations of the villages r at a time, and the $(n-r)^r \cdot F(n-r)$ modes having each of these sets for terminii, we have ${}_nC_r \cdot (n-r)^r \cdot F(n-r)$ modes, but in this reckoning each mode gets counted as many times as it has sets of r terminii, and hence one with exactly t terminii will get counted ${}_tC_r$ times. Therefore

$$\sum_{t} {}_tC_r \cdot g(n, t) = {}_nC_r \cdot (n-r)^r \cdot F(n-r). \qquad \ldots \quad (3.6)$$

a result which includes (3.4) and (3.5) as special cases.

4. Proofs of the Result $F(n) = n^{n-2}$.

When $n = 2$ there is only one way of connecting the two villages and $F(n) = 1 = 2^0$. Let us assume that the formula $F(n) = n^{n-2}$ is true for all values up to $n - 1$; we may then prove by using the

195326

currence formulae obtained above that the formula holds for the next value n also. Thus substituting in (3.1) we have

$$\sum_{r=0}^{r=2} \frac{(r+1)^{r-1}.(n-r-1)^{n-r-3}}{r!\,(n-r-2)!} = 2\,F(n)/n\,(n-2)! \ldots \quad (4.1)$$

To sum the series on the left, we make use of a theorem of Lagrange which enables us to expand x or any function of x in powers of y when x and y are connected by a relation of the form $x = y\,f(x)$. In particular if $x = y\,e^x$, we have an expansion for e^{ax} in the form*

$$e^{ax} = 1 + ay + a(a+2)\,y^2/2! + a(a+3)^2\,y^3/3! + \ldots \ldots$$

a series convergent for $|\,y\,| < 1/e$. Taking $a = 1$ and 2 respectively, we have in particular

$$e^x = 1 + y + 3\,y^2/2! + 4^2y^3/3! + \ldots \quad (r+1)^{r-1}\,y^r/r! + \ldots \quad (4.2)$$

and

$$e^{2x} = 1 + 2y + 8y^2/2! + 50y^3/3! \ldots + 2(r+2)^{r-1}\,y^r/r! + \ldots \quad (4.3)$$

Since the series (4.2) and (4.3) are absolutely convergent for sufficiently small values of y, we may square the former series and identify it with the latter. But the co-efficient of y^{n-2} in the square of the series for e^x is seen to be identical with the first member of the equality (4.1). Hence

$$2\,F(n)/n.(n-2)! = \text{co-efficient of } y^{n-2} \text{ in } (4.3)$$
$$= 2\,n^{n-3}/(n-2)!$$

whence it follows that $F(n) = n^{n-2}$.

Another proof of this result may be obtained by using (3.2) It is clear that the expression in (3.2) is merely the co-efficient of y^{n-r-1} in the expansion of the rth power of the series for e^x in (4.2) with the additional numerical factor $(n-1)!/r!$, *i.e.*, it is $(n-1)!$ times the co-efficient of y^{n-1} in $y^r e^{rx}/r!$. Hence by (3.3), $F(n)/(n-1)!$ is the co-efficient of

y^{n-1} in $\sum_{r=1}^{n-1}$ and hence also in $\sum_{r=1}^{\infty} y^r e^{rx}/r! = e^x$ since $x = ye^x$ $(n-1)!$.

* Bromwich: *Theory of Infinite Series.*

But by (4.2) this co-efficient is $n^{n-2}/(n-1)!$ whence it follows that $F(n) = n^{n-2}$.

Perhaps the simplest proof of the formula for $F(n)$ is the one which is based on the relations (3.6). Giving r the values 0, 1, 2, etc., we obtain a chain of equalities whence by alternate addition and subtraction we obtain

$$F(n) - n.(n-1)F(n-1) + {}_nC_2(n-2)^2.F(n-2)$$
$$- {}_nC_3(n-3)^3 F(n-3) + \ldots = 0,$$

since on the left $\qquad 1 - {}_tC_1 + {}_tC_2 - {}_tC_3 \ldots = 0.$

Substituting for $F(x)$ the value assumed x^{x-2} for all values of x up to $n-1$ we have;

$$0 = F(n) - n.(n-1)^{n-2} + {}_nC_2(n-2)^{n-2} + {}_nC_3(n-3)^{n-2} \ldots + .{}_nC_{n-1}\, 1^{n-2}$$

But it is a well-known result that*

$$0 = n^{n-2} - n.(n-1)^{n-2} + {}_nC_2(n-2)^{n-2} + \ldots + \ldots {}_nC_{n-1}\, 1^{n-2}$$

whence it follows by subtraction that

$$F(n) - n^{n-2} = 0.$$

5. The Function f(n, t).

It will be noticed that though (3.4) expresses $F(n)$ in terms of $f(n,t)$ it is through the elimination of these latter that $F(n)$ has been evaluated. The function $f(n, t)$ is in fact, rather a complicated function of n and t and its properties and structure will now be discussed.

We proceed to prove that

$$f(n, t) = (n-t).[f(n-1, t) + f(n-1, t-1)] \quad \ldots \quad (5.1)$$

Let the specified terminii be $v_1 v_2 \ldots v_t$.

---

* *Vide* Hall and Knight: *Higher Algebra*, or Chrystall: *Algebra*, Part II. It asserts that the expansion of

$$e^{nx} - n.e^{(n-1)x} + {}_nC_2 e^{(n-2)x} - \ldots + 1 = (e^x - 1)^n$$

does not contain a term in x^{n-2}.

Consider any one of the specified t terminii, say v_1. There is only one road through v_1 connecting it directly with, say v_r. Then v_r cannot itself be a terminus since in that case v_1 and v_r would together constitute an isolated system of two villages. There are two cases to be considered according as v_r is a junction of order two, or of a higher order. In the former case if v_1 and the road $v_1 v_r$ be destroyed, the remaining villages and roads form a simply connected system by (2.6) and (2.7), and this system has the villages $v_2 v_3 \ldots v_t$, and the village v_r for terminii. But when there are t specified terminii and these only for a simply connected system of $(n-1)$ villages, the number of possible modes of connection is $f(n-1, t)$. As v_r may be any of the villages $v_{t+1}, v_{t+2}, \ldots v_n$, it follows that there are $(n-t) f(n-1, t)$ possible simply connected systems of the first type. Consider next systems of the second type, *i.e.*, those in which v_r is of order higher than two. If v_1 and the road $(1r)$ be destroyed we have now a simply connected system of $n-1$ villages with the terminii $v_2 v_3 \ldots v_t$ and no others. Such a system may be obtained by forming all simply connected systems out of the $n-1$ villages other than v_1 with the terminii $v_2 v_3, \ldots v_t$, and connecting v_1 directly with any of he villages $v_{t+1} \ldots \ldots v_n$; hence the number of such systems is $(n-t) \,.\, f(n-1, t-1)$. The two types together make up all the $f(n, t)$ possible systems with the t specified terminii; hence the result (5.1).

The function $f(n, t)$ may be directly eveluated from its definition in certain simple cases. Thus by (2.5)

$$\left.\begin{aligned} f(n, 0) = f(n, 1) &= 0. \\ \text{And obviously} \qquad f(n, n-1) &= 1. \end{aligned}\right\} \qquad \ldots \quad (5.2)$$

Using these in association with (5.1), it is possible to obtain successively

$$\begin{aligned} f(n, 2) &= (n-2)! \\ f(n, 3) &= \tfrac{1}{2}(n-3)! \,.\, (n-2) \,.\, (n-3); \qquad \ldots \quad (5.3) \\ f(n, 4) &= (n-4)! \,.\, (n-4) \,.\, (n-3) \,.\, (n-2) \,.\, (3n-11) / 24, \text{ etc.} \\ f(n, n-2) &= 2^{n-2} - 2. \end{aligned}$$

For obtaining the numerical values of $(f(n, t)$ for small values of n and t, the simplest method would however, be to construct a table

like the one given bolow in which each entry is obtained by adding the number on its left and the one above the latter, and multiplying the sum by one for all entries in the leading diagonal, 2 for all those in the diagonal to its right, 3 for the next diagonal, etc. The figures in the leading diagonal are all units, and those in the first row are by (5.3) the factorials of the natural numbers. The entries are best made diagonal-wise.

Table of values of $f(n, t)$.

	$n = 3$	4	5	6	7	8	9	10
$t = 2$	1	2	6	24	120	720	5040	40320
3	0	1	6	36	240	1800	15120	141120
4	0	0	1	14	150	1560	16800	191520
5	0	0	0	1	30	540	8400	126000
6	0	0	0	9	1	62	1806	40824
7	0	0	0	0	0	1	126	5796
8	0	0	0	0	0	0	1	254
9	0	0	0	0	0	0	0	1
...	...	...	...	...	...	...	...	..

To express $f(n, t)$ explicitly in terms of n and t, we have to solve the functional equation (5.1) subject to the initial conditions (5.2).

Let $$f(n, t) = (n - t)!\ \phi(n, t) \qquad \ldots \qquad (5.4)$$

Substituting in (5 1) it is found that $\phi(n, t)$ satisfies

$$\phi(n, t) = \phi(n-1, t) + (n-t)\,.\,\phi(n-1), t-1) \qquad (5.5)$$

whence we have

$$\phi(n, t) = 1\,.\,\phi(t, t-1) + 2\,.\,\phi(t+1, t-1) + \ldots + (n-t)\,\phi(n-1, (t-1) \qquad \ldots \qquad (5.6)$$

Since by (5.3) and (5.4), $\phi(n, 1) = 0$, and $\phi(n, 2) = 1$, we have successively :

$$\phi(n, 3) = \sum_{1}^{n-3} r\,\phi(r+2, 2) = \sum_{r=1}^{n-3} r = \sigma_1(n-3) \text{ say.}$$

$$\phi(n, 4) = \sum_{1}^{n-4} r\,\phi(r+3, 3) = \sum_{1}^{n-4} r\,\sigma_1(r) = \sum_{1}^{n-4} n \sum_{1}^{n} n = \sigma_2(n-4), \text{ say}$$

$$\phi(n, 5) = \sum_{1}^{n-5} r\,\phi(r+4, 4) = \sum_{1}^{n-5} r\,\sigma_2(r) = \sum_{1}^{n-4} n \sum_{1}^{n} n \sum_{1}^{n} n = \sigma_3(n-5),$$

etc. ... *(5.7)

here being $t-2$ summations in the case of

$$\phi(n, t) = \sum_{1}^{n-t} n \sum n \sum n \ldots \sum n.$$

The formula (3.4) thus leads to the following curious identity :

$$\frac{1}{2!} + \frac{\sum_{1}^{n-3} n}{3!} + \frac{\sum_{1}^{n-4} n \sum n}{4!} + \frac{\sum_{1}^{n-5} n \sum n \sum n}{5!} \ldots \ldots = \frac{n^{n-2}}{n!} \ldots \quad (5.8)$$

6. Nomenclature.

It remains to choose a suitable method of representing the F (n) possible modes of connecting simply a set of n villages. Obviously, we may indicate the $n-1$ direct connections by $n-1$ bracket symbols of the form (rs) as in section 1, and these may again be combined into chains or routes $(rstu)$ ...) as indicated therein. Each chain may be extended so long as there are other villages which have direct connection with the villages at the ends of the chain. Hence we may suppose that each chain symbol like $(rst$ ...) has its end letters corresponding to terminii. If we have t terminii say 1, 2, ... t for a particular mode of simple connection, we obtain $t-1$ chains corresponding to the standard

---

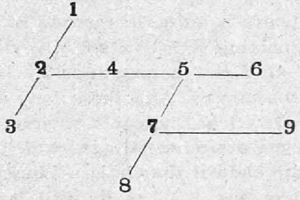
* An alternative form for $f(n, t)$ is

$$f(n, t) = \sum \frac{(n-2)!}{s_1!\, s_2! \ldots \ldots s_{n-t}!}$$

where $s_1, s_2, s_3, \ldots$ are integers $\not< 1$ and $s_1 + s_2 + \ldots s_{n-t} = n-2$, the summation extending to all possible sets of values of s satisfying the conditions. Cf. *J.I.M.S.*, Vol. 18, p. *191*.

routes from v_1 to v_2, v_3 ... v_t, respectively. It is also easy to see that these $t-1$ chains are quite adequate for indicating all the $n-1$ direct connections. For consider any particular direct connection say (rs). If either r or s is a terminus, say r, then the chain $(1 \ldots r)$ must involves (sr) as a last link as there is no other road through r. suppose that neither r nor s is a terminus. Then if (rs) be destroyed, the villages separate into 2 isolated systems (2,2), each of which is simply connected, one of which includes r and the other s. One of these (say the first) must have 1 for a terminus; and the other must have at least one terminus k other than s (if s is a terminus at all) by (2.5). Hence the standard route $(i \ldots k)$ involves the road (rs) since 1 and k belong to isolated systems if ths road be destroyed. As k is also one of the original t terminii, it follows that the direct connection (rs) is included as a link in one of the chains $(1 \ldots 2)$, $(1 \ldots 3)$, ... $(1 \ldots t)$. Hence these chains involve all the $n-1$ direct connections.

These $t-1$ chains may themselves be arranged in order in such way that the villages common to any two consecutive chains, are also common to all the chains thar follow. Thus the direct connections of the simply connected system represented in the figure will be represented chains :



(123) (12456) (124578) 124579).

There are 5 terminii and hence 4 chains. The representation is not however unique.

The canonical representation of a simply connected system discussed above may be made the basis for the calculation of $f(n-t)$. Thus if $t=2$, there is only one chain connecting the two terminii $(1 \ldots 2)$, and the other $n-2$ villages may be arranged in any order between 1 and 2; hence $f(n, 2) = (n-2)!$

If we have three assigned terminii 1, 2 and 3, we have two chains (1 ... 2) and (1 ... 3). If the common part of the two chains be deleted from the latter, we obtain a definite permutation of the $n - 3$ letters 4, 5, ... n, partitioned into two parts. Conversely from each of the $(n-3)!$ permutations of the letters, we form two chains by taking say the first p letters in order to form the middle of the first chain 1, and 2 being the extreme figures. For the second chain we write 1 at the beginning, follow it up by one or two or three ... or all the p letters of the first chain, and then write the remaining $n - 3 - p$ letters of the permutation in order, and finally 2. For a given p, the first chain can be formed in only one way while the second may be formed in p ways. Since p may take any value from 1 to $n - 3$, each permutation gives rise to $1 + 2 + \ldots n - 3$ possible chains (1 ... 2) and (1 ... 3). Hence $f(n, 3) = (n-3)!\,.\sum_{1}^{n-3} n.$

With four given terminii 1, 2, 3, and 4, we have three chains (1 ... 2), (1 ... 3), and (1 ... 4) and when the common part of the first two chains has been deleted from the second, and the common part of the second and third from the latter, we shall have a permutation of the $n - 4$ letters, 5, 6, ... n, partitioned into three parts. Conversely from any partition of these $n - 4$ letters we form all possible chains with the given terminii thus: Choose three numbers p, q, r such that

$$p + q + r = n - 4 \qquad p \geqslant 1,\ q \geqslant 0,\ r \geqslant 0. \qquad \ldots \quad (6.1)$$

Form the first chain with letter 1 at the beginning followed by the first p letters of the permutation in order, and with 2 at the end. For the second chain take any part* (but not the whole of the first chain from the beginning in order, and follow it up with the next q letters of the permutation. Since the point of departure of the second chain may be any of the p letters of the first chain, it may be formed in p ways. For the third chain take any part (but not the whole) of either of the two chains formed already, and add the next r letters of the permutation, which will now be exhausted by (6.1). As the point of departure of the third chain from either of the first two may be any of the $p + q$ inter-

* The part should consist of at least two letters, since otherwise 1 would become a junction.

mediate letters, it can be formed in $p+q$ ways for every way of forming the first two chains. Hence when the permutation and the numbers p, q, r are given, there are $p(p+q)$ ways of forming the three chains. Allowing p and q to take all possible values consistent with (6.1), we have

$$f(n, 4) = (n-4)!\ \Sigma\ \Sigma\ p \cdot p' \qquad [p \leqslant p' \leqslant n-4]$$

$$= \left(\sum_{1}^{n-4} p' \sum_{1}^{p'} p\right) \times (n-4)!$$

$$= (n-4)!\ \phi(n, 4) \qquad (\textit{vide } \mathbf{5.7})$$

since there are $(n-4)$ possible permutations.

It is not difficult to prove by similar reasoning that when all possible chains have been formed with t terminii their number is

$$(n-t)!\,.\,\Sigma\ p \cdot p' p'' \cdot p''' \ldots p^{t-2} \qquad [p \leqslant p' \leqslant p'' \leqslant \ldots p^{(t-2)} \leqslant n-t]$$

and that this is identical with

$$(n-t)!\,.\,\sum_{1}^{n-t} n \sum_{1}^{n} n \sum_{1}^{n} n \ldots \sum_{1}^{n} n$$

so that the method of enumeration by nomenclature does not lead to a new method of evaluating F (n), but rather furnishes an alternative proof of the formula for $f(n, t)$ obtained already, and hence leads ultimately to the identity (5.8). It enables us however, to write down all the possible modes of direct connection.

7. Equivalent Problem in Group Theory.

Let us associate with each direct connection $(rs) = 1$, the transposition (rs) and consider the substitution group in n letters generated by all the transpositions which correspond to direct connections which make up a simply connected system. Now from the transposition (rs) and (st) we may obtain $(rt) = (rs)\,(st)\,(rs)$, which therefore belongs to the group. Since any two letters a, b are connected by a chain of direct connections $(a \ldots b) = 1$. it follows that the group contains the transpositions (ab). And as any arbitrary permutation can be expressed as the product of transpositions only, it is clear that the group has an operator which transforms any permutation into any other, and is therefore the symmetric group of degree n. We have therefore proved that

the symmetric group of degree n may be generated in $F(n)$ different ways by means of $n-1$ transpositions. ... (7.1)

Or again,

if t assigned letters are to figure, each in one transposition only, then the number of ways of generating the symmetric group in n letters is $f(n, t)$.

All our propositions may now be given an interpretation in group theory, a junction of order k meaning that the particular letter shall appear in exactly k of the generating transposition.

8. The Equivalent Problem in Matrix Algebra.

Let us mean by a positive square matrix, a matrix with an equal number of rows and columns all of whose elements are positive or zero. In this section unless the contrary is stated, all the matrices considered are positive square matrices.

By an *elementary transformation*, we shall mean a transformation of the elements of a matrix, in which each is multiplied by a positive non-zero constant, not necessarily the same for all elements. Two matrices of which one can be transformed into the other by means of an elementary transformation are *equivalent*. It is obvious that equivalence is a symmetric property and is transitive, We shall indicate equivalence by the symbol "$=$." It is also clear that any matrix is equivalent to another in which the elements are either 0's or 1's, and that all equivalent matrices correspond to the same canonical matrix of the type just indicated.

The law of addition is the usual law, *i.e.*, corresponding elements are added in the case of square matrices of the same order which is the only case we consider here. It is clear that with our definition of equivalence we have ... (8.1)

$$A + A = A.$$

Let us write $A > B$, if whenever B has a non-zero element, A has also a corresponding non-zero element, while for at least one zero element of B there corresponds a non-zero element of A. It is clear that if $A > B$ and $B > C$, then $A > C$, and that if $A > B$ and $B = C$, then $A > C$.

The matrix all of whose elements are zero, we denote by O and the matrix all of whose elements are 1 by I. It is clear that

$$\mathrm{I} \geqslant \mathrm{A} \geqslant \mathrm{O} \qquad \text{whatever A.} \qquad \ldots \quad (8.2)$$

We define multiplication in the usual way, that is, A . B = C

where $\mathrm{A} = [a_{rs}]$, $\mathrm{B} = [b_{rs}]$, $\mathrm{C} = [c_{rs}]$

and $c_{rs} = \sum_t a_{rt}\, b_{ts}$.

We denote the matrix all of whose elements in the leading diagonal are 1's and all the other elements zero, by E. Then we have

$$\mathrm{A}.\mathrm{E} = \mathrm{E}.\mathrm{A} = \mathrm{A} \quad \text{whatever A.}$$

A matrix M which satisfies the inequality $\mathrm{M} \geqslant \mathrm{E}$ is called a *uniaxial* matrix and has the property that all its elements in the leading diagonal are different from zero.

We now prove that if A, B, A′, B′, be uniaxial, and if $\mathrm{A} > \mathrm{A}'$ and $\mathrm{B} > \mathrm{B}'$, then $\mathrm{A}.\mathrm{B} \geqslant \mathrm{A}'.\mathrm{B}'$. ... (8.3)

This follows from the law of multiplication of matrices for if $\mathrm{AB} = \mathrm{C}$ and $\mathrm{A'B'} = \mathrm{C'}$, then $c_{rs} = \Sigma\, a_{rt}\, b_{ts}$ and $c'_{rs} = \Sigma\, a'_{rt}\, b'_{ts}$ and $a_{rt} \geqslant a'_{rt}$ and $b_{ts} \geqslant b'_{ss}$* since $\mathrm{A} > \mathrm{A}'$ and $\mathrm{B} > \mathrm{B}'$.

In particular if A be uniaxial, so that $\mathrm{A} \geqslant \mathrm{E}$, we have $\mathrm{A}^2 \geqslant \mathrm{AE} = \mathrm{A}$; $\mathrm{A}^3 \geqslant \mathrm{A}^2$; and so on. If at any stage we have $\mathrm{A}^k = \mathrm{I}$ then we have $\mathrm{A}^{k+1} = \mathrm{A}^{k+} = \mathrm{A}^{k+3} = \ldots = \mathrm{I}$.

One problem of interest in this branch of matrix algebra is to find all the solutions of the matrix equation

$$\mathrm{A}^k = 1 \qquad \ldots \quad (8.4)$$

where A is a matrix with n rows and n colums, and k is a given positive integer. Two solutions which are equivalent may be treated as identical so that we may confine ourselves to matrices whose elements consist of

---

* All the matrices are supposed to be in their equivalent canonical form so that the elements are either 1 or zero.

0's and 1's. Again from (8.3), if A satisfies (8.4), and B > A, then $B^k \geqslant I$. $\therefore B^k = I$ so that B also satisfies the equation. It is therefore sufficient to confine ourselves to what we may call the ***primitive solutions*** A, namely those for which it is not true that $C^k = I$, $C < A$.

Now between E, A, A^2, A^3 ... A^n there is a linear relation (the Cayley-Hamilton equation) of the shape

$$c_0 E + c_1 A + c_2 A^2 + \dots \dots c_n A^n \equiv O.$$

where the c's are scalars so that if the element in the rth row and the sth column in E, A, A^2 ... A^{n-1} be all zeros, then the corresponding element in A^n is also zero. It may be easily shown that the corresponding element in A^{n+1}, A^{n+2}, etc., will also be zero, so that no power of A can be equal to I.

Let us confine ourselves to uniaxial solutions of (8.4).

Since, in this case, A > E, It follows that $A^{n-1} \geqslant A^r$ for all values of r less than $n-1$. Hence if A^{n-1} has a zero element anywhere the corresponding element in all lower powers of A are all zero, and hence no power of A can be equivalent to I. Hence we infer that

All uniaxial solutions of (8.4) are also solutions of $A^{n-1} = I$. (8.5)

I shall now show that this problem of finding primitive uniaxial solutions of (8.4) is identical with the topological problem considered in the paper.

For, consider some one method of connecting the n viilages simply by $n-1$ roads. Construct a matrix A of the nth order in which $a_{rr} = 1$ for all r, and $a_{rs} = (rs)$ that is, $a_{rs} = 1$ if v_r and v_s are directly connected and $a_{rs} = 0$ if they are not. Consider the canonical or m of $A^2 = B$. Since A is uniaxial, B > A, so that $b_{rs} = 1$ if r and s are directly connected. It will also be 1, if r and s are both directly connected with another village t, for $b_{rs} = \sum_t a_{rt}\, a_{ts}$ and for a certain t we have $a_{rt} = 1$ and $a_{ts} = 1$. Similarly A^3 will contain 1 in the rth row and sth column, if it is possible to pass from v_r to v_s by a chain of three links like $(rlms) = 1$; for it is $\sum_{l,m} a_{rl} \cdot a_{lm} \cdot a_{ms}$ and if there exist

path from r to s with two intermediate villages we have $a_{rl} = 1$, $a_{lm} = 1$. $a_{ms} = 1$. And generally the canonical form A^k will have the unit element whenever it is possible to pass from the village corresponding to the row to the village corresponding to the number of the column by k or a smaller number of steps. The result (8,5) now becomes obvious, for, in a simply connected system, which corresponds to a primitive solutions, there are $n - 1$ roads, and there is a standard route from any village to any other which will not require more than $n - 1$ roads, *i.e.*, $n - 2$ intermediate villages altogether, We therefore infer that *there are* n^{n-2} *solutions (all non-equivalent and primitive) of the equation* $A^{n-1} = 1$.

It is a matter of interest to examine also, the zero axial matrices in which the element $a_{rs} = (rs) = 1$ or 0 according as v_r and v_s are, or are not directly connected in a simply connected system of n villages. If A^2 (the square of the zero axial matrix) = B, then $b_{rs} = \Sigma\ (rts)$ as defined in section 1, and will be 0 or otherwise according as it is impossible to pass from r to s in two steps, or possible to do so. Now if r and s are directly connected, the Σ contains one term $a_{rs} . a_{ss}$, but, unlike the case of a uniaxial matrix, $a_{ss} = 0$, so that $a_{rs} = 1$ does not ensure that b_{rs} is non-zero. On the other hand if $(rs) = 1$ in a simply connected system, there cannot exist a t directly connected with both r and s, so that the matrix A^2 will have a zero element corresponding to each direct connection. Similarly if $A^3 = C$, then $c_{rs} = \Sigma\ (rlms)$ and will be zero unless it is possible to pass from r to s in *exactly* three steps, retracing the same step being however permitted. In fact for every direct connection $(rs) = 1$, we shall have $c_{rs} = 1$,* since if s is directly connected with t, we have $(rs)\ (st)\ (ts) = 1$ is one term in the summation for c_{rs}. And generally A^k will have a non-zero r, sth element if it is possible to pass from r to s in exactly k steps, including steps traced more than once. If A be a zero axial matrix which arises from a simply connected system of n villages, then it is possible to pass from any village

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* As will have been already noticed, "1" is used in this paper often in the sense of "not zero" A non-zero element becomes replaced by 1 when it is reduced to a canonical form by an elementary transformation.

to any other in 1 or 2 ... or $n-1$ steps, so that the r, sth element either in A or A^2 or ... A^{n-1} must be different from zero, or in symbols:

$$E + A + A^2 + A^3 \ldots A^{n-1} = I.$$

This is consistent with the result (8 5) previously obtained for uniaxial matrices corresponding to a simply connected system, for any such matrix may be written in the form $A + E$ where A is zero-axial. Hence by 8.5,

$$(A + E)^{n-1} = A^{n-1} + (n-1) A^{n-2} E + {}_{n-1}C_2 A^{n-3} E^2 + \ldots E^{n-1}$$

$$= A^{n-1} + A^{n-2} + A^{n-3} \ldots A + E = I$$

Since $E . X = X$, and $X + X = X$ whatever X.

It is a problem of considerable interest to find out the number of non-equivalent solutions of $A^k = I$ where $k < n-1$. If the solutions are uniaxial and symmetric, each of them determines one method of connecting n villages by $n-1$ roads so that it will be possible to pass from any one to any other by a route involving k roads or less.

THE THEORY OF INTERMITTENT ACTION AND BAND-SPECTRA IN INFRA-RED.

By D. K. Sen and B. N. Biswas.

The theory of intermittent action has been developed in a series of papers.* by Kar and others. The problem of Band-Spectra in Infra-red has been solved with the help of this theory by Kar and Biswas in a paper in *Zeitschrift für Physik*, 1930, Bd. 59, 570. In the present paper we have used a different method to solve the problem and have further obtained the modifications for the energy-expression when the terms in x^3 and x^4 are included in the differential equation of motion of the anharmonic oscillator.

1. According to this theory the radiation acts on the oscillator intermittently and imparts to the latter equal energy $= h\nu_0$ at each interaction, where h is Planck's constant. Let $U_1, U_2, \ldots U_n$ denote the velocities of the oscillator just after 1st, 2nd ... *nth* impulse.' Then

$$\begin{aligned}
\tfrac{1}{2} m U_1^2 &= h\nu_0, \\
\tfrac{1}{2} m U_2^2 - \tfrac{1}{2} m U_1^2 &= h\nu_0, \\
\tfrac{1}{2} m U_3^2 - \tfrac{1}{2} m U_2^2 &= h\nu_0, \\
\cdot \quad \cdot \quad \cdot \quad & \cdot \quad \cdot \\
\cdot \quad \cdot \quad \cdot \quad & \cdot \quad \cdot \\
\tfrac{1}{2} m U_n^2 - \tfrac{1}{2} m U_{n-1}^2 &= h\nu_0.
\end{aligned}$$

Adding, we get

$$\tfrac{1}{2} m U_n^2 = n h\nu_0 \quad \ldots \quad (1)$$

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* K. C. Kar : *Phys. Zs.*, 24, 63, 1923 ; *Phys. Rev.* 21, 695, 1923 ; *Zs. f. Phys.* (in Press)

K. C. Kar and M. Ghosh : *Phys. Zs.*, 29, 143, 1928 ;

M. Ghosh : *Phys. Zs.* 30. 160, 1929.

K. C. Kar and R. C. Majumdar : *Zs. f. Phys.*, 55, 546, 1929 ; *Zs. f. Phys.* 53, 308, 1929 ;

K. C. Kar and B . N. Biswas : *Zs. f. Phys.*, 59, 570. 1930.

B. N. Biswas : *Journal I. M. Soc.*, XVIII, No. 8, 1930.

The equation of notion of the harmonic oscillator is

$$\ddot{x} + \lambda^2 x = 0. \qquad \ldots \quad (2)$$

where $\lambda = 2\pi\nu_0$ Its solution is

$$x = \frac{U_n}{\lambda} \sin \lambda t = x_0 \text{ (say)} \qquad \ldots \quad (3)$$

for motion after nth impulse; so that

$$\dot{x}_0 = U_n \cos \lambda t \qquad \ldots \quad (4)$$

∴ Energy of the oscillator = sum of potential and kinetic energies

or
$$E(n) = \tfrac{1}{2} m\lambda^2 x_0^2 + \tfrac{1}{2} m\dot{x}_0^2 = \tfrac{1}{2} m U_n^2 = n h \nu_0 \qquad \ldots \quad (5)$$

from (1).

2. Let x and $\dot{x}$ denote the position and velocity at any instant, of the anharmonic oscillator up to a certain approximation, and ξ and $\dot{\xi}$ denote changes in the above quantities for a further degree of approximation. Let P and K denote the additional parts of the potential and kinetic energies for the second approximation. As, in determining the energy of the nth state of the oscillator we have to deal with mean values, we make the following conventions :

(i) Additional Energy-term or $E = \pm \overline{P} \pm \overline{K}$; ... (6)

(ii) $\overline{P}$ is to be taken positive or negative according as $\overline{\xi}$ is positive or negative;

(iii) $\overline{K}$ is to be taken positive if $\overline{\dot{\xi}}$ is 0.

3. The equation of motion of the anharmonic oscillator is

$$\ddot{x} + \lambda^2 x = -ax^2 - bx^3 - cx^4 \ldots \ldots \qquad (7)$$

where $a, b, c, \ldots \ldots$ are quantities of increasing orders of smallness. Taking only one term, at first, on the right-hand side we write (7) as

$$\ddot{x} + \lambda^2 x = -ax^2 = -\frac{aU_n^2}{\lambda^2} \sin^2 \lambda t$$

$$= -\frac{aU_n^2}{2\lambda^2}(1 - \cos 2\lambda t) \qquad \ldots \quad (8)$$

substituiing on the right-hand side from (3). The solution of thie equation satisfying the initial conditions, $x = 0$, $\dot{x} = U_n$, is

$$x = \frac{U_n}{\lambda} \sin \lambda t - \frac{a\, U_n^2}{2\, \lambda^4} \left(1 - \frac{4}{3} \cos \lambda t + \frac{1}{3} \cos 2\, \lambda t\right) = x_1 \text{ (say)} \qquad \dots \quad (9)$$

Here, deviation being regarded from the harmonic oscillator, we have

$$\xi = -\frac{a\, U_n^2}{2\, \lambda^4} \left(1 - \frac{4}{3} \cos \lambda t + \frac{1}{3} \cos 2\, \lambda t\right) \qquad \dots \quad (10)$$

and correspondiug expression for $\dot{\xi}$.

$$P = \tfrac{1}{2}\, m\, \lambda^2 (x_1^2 - x_0^2)$$

$$= -\frac{m a\, U_n^3}{2\, \lambda^3} \sin \lambda t \left(1 - \frac{4}{3} \cos \lambda t + \frac{1}{3} \cos 2'\, \lambda t\right)$$

$$+ \frac{m a^2\, U_n^4}{8\, \lambda^6} \left(1 - \frac{4}{3} \cos \lambda t + \frac{1}{3} \cos 2\, \lambda t\right)^2$$

$$\therefore \quad \overline{P} = \frac{m a^2\, U_n^4}{8\, \lambda^6} \left(1 + \frac{16}{18} + \frac{1}{18}\right) = \frac{35}{144} \frac{m a^2\, U_n^4}{\lambda^6} \qquad \dots \quad (11)$$

And $\quad K = \tfrac{1}{2}\, m\, (\dot{x}_1^2 - \dot{x}_0^2)$

$$= -\frac{m a\, U_n^3}{2\, \lambda^3} \sin \lambda t \left(\frac{4}{3} \sin \lambda t - \frac{2}{3} \sin 2\, \lambda t\right)$$

$$+ \frac{m a^2\, U_n^4}{8\, \lambda^6} \left(\frac{4}{3} \sin \lambda t - \frac{2}{3} \sin 2\, \lambda t\right)^2.$$

$$\therefore \quad \overline{K} = \frac{m a^2\, U_n^4}{8\, \lambda^6} \left(\frac{16}{18} + \frac{4}{18}\right) = \frac{20}{144} \frac{m a^2\, U_n^4}{\lambda^6} \qquad \dots \quad (12)$$

Now $\overline{\xi} = -\dfrac{a\, U_n^2}{2\, \lambda^4}$, *i.e.*, it is negative and $\overline{\dot{\xi}}$ is zero. Therefore

$$E = -\overline{P} + \overline{K} = -\frac{15}{144} \frac{m a^2\, U_n^5}{\lambda^6}. \qquad \dots \quad (13)$$

so that for the anharmonic oscillator we have

$$E(n) = n h \nu_0 - \frac{15}{144} \frac{m a^2 U_n^4}{\lambda^6}. \qquad \dots \quad (14)$$

Putting $\frac{1}{2} m U_n^2 = n h \nu_0$ from (1), $\lambda = 2\pi \nu_0$ and $ma = 3a_0$, we get the well-known expression*

$$E(n) = n h \nu_0 - \frac{15}{32} \frac{a_0^2 h^2 n^2}{(2\pi^2 \nu_0 m)^3}. \quad \ldots \quad (15)$$

4. ***Second Approximation.*** We take one more term on the right-hand side, that is, our equation becomes

$$\ddot{x} + \lambda^2 x = -ax^2 - bx^3. \quad \ldots \quad (16)$$

As we do not at the present stage require any term in the solution involving the product ab, we may consider the simpler equation

$$\ddot{x} + \lambda^2 x = -bx^3, \quad \ldots \quad (17)$$

and add (10) or (13) to the corresponding results that will be deduced from this equation. Putting $x = \frac{U_n}{\lambda} \sin \lambda t$ on the right-hand side we get

$$\ddot{x} + \lambda^2 x = -\frac{bU_n^3}{\lambda^3} \sin^3 \lambda t = -\frac{bU_n^3}{4\lambda^3}(3 \sin \lambda t - \sin 3\lambda t)$$

$$= -\frac{3}{4} \frac{bU_n^2}{\lambda^2} x + \frac{1}{4} \frac{bU_n^3}{\lambda^3} \sin 3\lambda t. \quad \ldots \quad (18)$$

up to our approximation. Transposing the term in x to the left-hand side and putting

$$\lambda_1^2 = \lambda^2 + \frac{3}{4} \frac{bU_n^2}{\lambda^2} \quad \ldots \quad (19)$$

we obtain

$$\ddot{x} + \lambda_1^2 x = \frac{1}{4} \frac{bU_n^3}{\lambda^3} \sin 3\lambda t. \quad \ldots \quad (20)$$

The solution of (20) satisfying the initial conditions, $x = 0$, $\dot{x} = U_n$ is

$$x = \frac{U_n}{\lambda_1} \sin \lambda_1 t + \frac{bU_n^3}{\lambda^5} \left(\frac{3}{32} \sin \lambda_1 t - \frac{1}{32} \sin 3\lambda t \right) ; \quad \ldots \quad (21)$$

this will give corresponding expression for $\dot{x}$.

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* *Quantum Theory of the Atom*, by Birtwistle, p. **216**,

We have from (19)

$$\lambda_1^2 = \lambda^2\left(1 + \frac{3}{4}\,\frac{b\,U_n^2}{\lambda^4}\right)$$

whence $$\lambda_1 = \lambda\left(1 + \frac{3}{8}\,\frac{b\,U_n^2}{\lambda^4}\right) = \lambda\,(1 + \varepsilon),$$

say, up to this approximation. In this case

$$P = \tfrac{1}{2}\,m\lambda^2\,(x^2 - x_0^2)$$

$$= \tfrac{1}{2}\,m\lambda^2\left[\frac{U_n^2}{\lambda_1^2}\sin^2\lambda_1 t + \frac{2b\,U_n^4}{\lambda_1\lambda^5}\left(\frac{3}{32}\sin^2\lambda_1 t - \frac{1}{32}\sin\lambda_1 t\,\sin 3\lambda t\right) - \frac{U_n^2}{\lambda^2}\sin^2\lambda t\right] \quad \ldots \quad (22)$$

Now $\sin\lambda_1 t$ in (21) is equivalent, to two terms, one in $\sin\lambda t$ and the other in $\cos\lambda t$, the latter being due to the extra term (in b) in the differential equation of motion And $\cos\lambda t$ is zero for $\lambda t = \frac{\pi}{2}, \frac{5\pi}{2}, \ldots\ldots$

Therefore the limits of integration are to be taken as $\frac{\pi}{2\lambda}$ and $\frac{5\pi}{2\lambda}$ in evaluating the mean values of the terms involving $\lambda_1 t$. We have then

$$\overline{\cos\lambda_1 t} = 0;\ \overline{\sin\lambda_1 t} = \varepsilon \text{ and } \overline{\cos 2\lambda_1 t} = -\varepsilon. \quad \ldots \quad (23)$$

$$\therefore\ \overline{P} = \tfrac{1}{2}\,m\left[-\frac{3}{16}\,\frac{b\,U_n^4}{\lambda^4} + \frac{b\,U_n^4}{\lambda^4}\left(\frac{3}{32}\right)\right] = -\frac{3}{64}\,\frac{mb\,U_n^4}{\lambda^4} \quad \ldots \quad (24)$$

Similarly

$$\overline{K} = -\frac{3}{64}\,\frac{mb\,U_n^4}{\lambda^4} \quad \ldots \quad (25)$$

Now, $\overline{\xi} = \overline{x - x_0}$, is positive as $\overline{\sin\lambda_1 t}$ is positive; and $\overline{\xi}$ is 0.

$$\therefore\quad E = -\overline{P} - \overline{K} = \frac{3}{32}\,\frac{mb\,U_n^4}{\lambda^4}, \quad \ldots \quad (26)$$

from (24) and (25).

$$\therefore\ E\,(n) = nh\nu_0 - \frac{15}{32}\,\frac{a_0^2h^2n^2}{(2\pi^2\nu_0 m)^2} + \frac{3}{8}\,b_0\cdot\left(\frac{hn}{2\pi^2\nu_0 m}\right)^2 \quad \ldots \quad (27)$$

where $mb = 4b_0$.*

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* *Quantum Theory of the Atom l. c.*

5. In (11) and 12) we have taken terms involving a^2 into consideration, whilst in (8) and (9) we have considered terms only up to a. If we now include such terms we may write the equation of motion as

$$\ddot{x} + \lambda^2 x = -ax^2$$
$$= -a\left[\frac{U_n}{\lambda}\sin\lambda t - \frac{aU_n^{\,2}}{2\lambda^4}\left(1-\frac{4}{3}\cos\lambda t + \frac{1}{3}\cos 2\lambda t\right)\right]^2$$

from (9).

$$= -\frac{aU_n^{\,2}}{2\lambda^2}\left(1-\cos 2\lambda t\right) + \frac{a^2 U_n^{\,3}}{\lambda^5}\left(\frac{5}{6}\sin\lambda t - \frac{2}{3}\sin 2\lambda t + \frac{1}{6}\sin 3\lambda t\right) \quad \ldots \quad (28)$$

Now, in the second bracket we may replace $\frac{U_n}{\lambda}\sin\lambda t$ by x, since there is the factor a^2; transposing this term to the left-hand side and putting

$$\lambda_2^{\,2} = \lambda^2 - \frac{5}{6}\frac{a^2 U_n^{\,2}}{\lambda^4},$$

we get

$$\ddot{x} + \lambda_2^{\,2}x = -\frac{aU_n^{\,2}}{2\lambda^2}\left(1-\cos 2\lambda t\right)$$
$$+ \frac{a^2 U_n^{\,3}}{\lambda^5}\left(-\frac{2}{3}\sin 2\lambda t + \frac{1}{6}\sin 3\lambda t\right) \quad \ldots \quad (30)$$

whence

$$x = \frac{U_n}{\lambda_2}\sin\lambda_2 t - \frac{aU_n^{\,2}}{2\lambda^4}\left(1-\frac{4}{3}\cos\lambda_2 t + \frac{1}{3}\cos 2\lambda t\right)$$
$$+ \frac{a^2 U_n^{\,3}}{\lambda^7}\left(-\frac{55}{144}\sin\lambda_2 t + \frac{2}{9}\sin 2\lambda t - \frac{1}{48}\sin 3\lambda t\right) \quad \ldots \quad (31)$$

which satisfies the initial conditions. Proceeding as in the last article, we obtain

$$\overline{P} = \left(\frac{5}{48} + \frac{35}{144} - \frac{55}{288}\right)\frac{ma^2 U_n^{\,4}}{\lambda^6} = \frac{45}{288}\frac{ma^2 U_n^{\,4}}{\lambda^6}, \quad \ldots \quad (32)$$

$$\text{and}\quad \overline{K} = \left(\frac{5}{48} + \frac{20}{144} - \frac{55}{288}\right)\frac{ma^2 U_n^{\,4}}{\lambda^6} = \frac{15}{288}\frac{ma^2 U_n^{\,4}}{\lambda^6} \quad \ldots \quad (33)$$

Also $\overline{\xi} = -\frac{aU_n^{\,2}}{2\lambda^4} - \frac{5}{12}\frac{a^2 U_n^{\,3}}{\lambda^8}$, *i.e.*, is negative and $\overline{\overline{\xi}}$ is zero.

$$\therefore\quad E = -\overline{P} + \overline{K} = -\frac{15}{144}\frac{ma^2 U_n^{\,4}}{\lambda^6},$$

as in (13).

6. We proceed to next approximation up to a^3. The equation (28) now becomes

$$\ddot{x} + \lambda^2 x = -\frac{a\,U_n^{\,2}}{2\,\lambda^2}(\dots\ \dots) + \frac{a^2\,U_n^{\,3}}{\lambda^5}(\dots\ \dots)$$

$$-\frac{a^3\,U_n^{\,4}}{4\,\lambda^8}\left(\frac{25}{12} - \frac{20}{9}\cos\lambda t + \frac{4}{3}\cos 2\,\lambda t - \frac{4}{3}\cos 3\,\lambda t + \frac{5}{36}\cos 4\,\lambda t\right) \qquad \dots \quad (34)$$

where (... ...) stand for the corresponding expressions in (28). If we put

$$\lambda'^2 = \lambda^2\left(1 - \frac{5}{9}\,\frac{a^3\,U_n^{\,4}}{\lambda^{10}}\right) \qquad \dots \quad (35)$$

we see that

$$\left(\frac{d^2}{dt^2} + \lambda^2\right)\cos\lambda' t = \frac{5}{9}\,\frac{a^3\,U_n^{\,4}}{\lambda^8}\cos\lambda' t = \frac{5}{9}\,\frac{a^3\,U_n^{\,4}}{\lambda^8}\cos\lambda t$$

to this degree of approximation. Therefore, the solution of (34) may be written as

$$x = \text{right-hand side of (31)} + (\cos\lambda' t - \cos\lambda t)$$

$$-\frac{a^3\,U_n^{\,4}}{4\,\lambda^{10}}\left(\frac{25}{12} - \frac{97}{54}\cos\lambda t - \frac{4}{9}\cos 2\,\lambda t + \frac{1}{6}\cos 3\,\lambda t - \frac{1}{108}\cos 4\,\lambda t\right) \qquad \dots \quad (36)$$

It can be seen that the extra terms of (36) do not contribute anything to the expression for energy.

7. If we include terms in (ab), the equation of motion becomes

$$\ddot{x} + \lambda^2 x = \text{right-hand side of (34)}$$

$$+\frac{ab\,U_n^{\,4}}{\lambda^6}\left(\frac{29}{32} - \frac{1}{2}\cos\lambda t - \frac{5}{4}\cos 2\lambda t + \frac{1}{2}\cos 3\lambda t - \frac{5}{32}\cos 4\lambda t\right) \qquad \dots \quad (37)$$

We can obtain the solution of this as in the last article and obtain

$$x = \text{right hand side of (31)} + (\cos\lambda_1' t - \cos\lambda t) - \frac{a^3 U_n^{\,4}}{4\lambda^{10}}(\dots\ \dots)$$

$$+\frac{ab\,U_n^{\,4}}{\lambda^8}\left(\frac{29}{32} + \frac{5}{12}\cos 2\lambda t - \frac{1}{16}\cos 3\lambda t + \frac{1}{96}\cos 4\lambda t\right) \qquad \dots \quad (38)$$

where (... ...) stands for the corresponding expression in (36), and

$$\lambda'^{2}_{1} = \lambda^{2}\left(1 - \frac{5}{9}\,\frac{a^{3}U_{n}^{4}}{\lambda^{10}} + \frac{1}{2}\,\frac{abU_{n}^{4}}{\lambda^{8}}\right). \qquad \ldots \quad (39)$$

E due to the additional term will be seen to be

$$\frac{\pi}{4}\,\frac{mabU_{n}^{5}}{\lambda^{7}}. \qquad \ldots \quad (40)$$

This process can be continued up to any degree of approximation.

8. Lastly, we introduce the term $-cx^4$ on the right-hand side of the equation of motion. As we do not propose to bring in any terms involving the products of c with a, b or c, we shall consider simply the equation

$$\ddot{x} + \lambda^{2}x = -cx^{4}. \qquad \ldots \quad (41)$$

and add the solution to those obtained in previous articles. On the right-hand side of this equation we put $x = x_0$ and we obtain

$$\ddot{x} + \lambda^{2}x = -\frac{cU_{n}^{4}}{\lambda^{4}}\sin^{4}\lambda t$$

the solution of which is

$$x = -\frac{cU_{n}^{4}}{\lambda^{6}}\left(\frac{3}{8} - \frac{8}{15}\cos\lambda t + \frac{1}{6}\cos 2\lambda t - \frac{1}{120}\cos 4\lambda t\right) \qquad \ldots \quad (42)$$

satisfying the initial conditions $x = 0$, $\dot{x} = 0$. This additional term for x does not add any term to $E(n)$, the expression for energy.

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A THEOREM CONCERNING INTEGRAL FUNCTIONS OF ORDER UNITY AND TYPE POSITIVE.

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1. Let $\phi(z) = \sum_0^\infty \frac{c_n z^n}{n!}$ be an integral function of $z \, (= \rho e^{i\psi})$, not identically zero, and suppose $\phi(z)$ satisfies the relation

$$\phi(z) = O(e^{M\rho}), \quad \ldots \quad (1.1)$$

where M is a constant, throughout the plane. Let

$$\lambda(\psi) \equiv \overline{\lim_{\rho \to \infty}} \frac{\log |\phi(\rho e^{i\psi})|}{\rho}. \quad \ldots \quad (1.2)$$

Then it is evident that $\lambda(\psi)$ can not always be negative, and we also know that in order that the maximum value of $\lambda(\psi)$ in the range $-\pi \leqslant \psi \leqslant \pi$ be positive, it is both necessary and sufficient that the Taylor's series $\sum_0^\infty c_n z^n$ should have a finite radius of convergence. Now one may naturally ask: what is the necessary and sufficient condition in order that the minimum value of $\lambda(\psi)$ in the range $|\psi| \leqslant \pi$ should be negative? The answer to this question is supplied by the following theorem, which it is the main object of this paper to establish.

THEOREM.—*If*

$\phi(z) = \sum_0^\infty \frac{c_n z^n}{n!}$ be an integral function of order 1 and type positive; ... (1.3)

the minimum value of $\lambda(\psi)$ in the range $|\psi| \leqslant \pi$ be negative, then there exists (and in fact uniquely) an integral function $c(z)$ satisfying the following conditions; ... (1.4)

$c(z)$ is an integral function of order 1 and type $l \geqslant 0$; ... (1.5)

$$c(n) = c_n, \qquad (n = 0, 1, 2, \ldots): \qquad \ldots \quad (1.6)$$

$$0 \leqslant H\left(\frac{\pi}{2}\right) + H\left(-\frac{\pi}{2}\right) < \pi, \qquad \ldots \quad (1.7)$$

where H (ψ) *is formed from* $c(z)$ *as* $\lambda(\psi)$ *is from* $\phi(z)$*; and conversely.*

2. In this section we shall establish the direct theorem.

$\lambda(\psi)$ is a continuous function of ψ in the range $|\psi| \leqslant \pi$, and, if k is the type of $\phi(z)$, $-k \leqslant \lambda(\psi) \leqslant k$, the value k being attained for at least one value of ψ. Also $\lambda(\psi)$ is a periodic function with a period of 2π, and satisfies the inequality

$$\lambda(\psi_1)\sin(\psi_3 - \psi_2) + \lambda(\psi_2)\sin(\psi_1 - \psi_3) + \lambda(\psi_3)\sin(\psi_2 - \psi_1) \geqslant 0,$$

ψ_1, ψ_2, ψ_3 being any three values of ψ which satisfy the inequalities ... (2.1)

$$\psi_1 < \psi_2 < \psi_3, \quad \psi_2 - \psi_1 < \pi, \quad \psi_3 - \psi_2 < \pi. \qquad \ldots \quad (2.2)$$

Since the minimum value of $\lambda(\psi)$ is negative, there exists an interval of ψ, at most equal to π in extent, inside which $\lambda(\psi)$ is negative, and which is bounded by values of ψ for which $\lambda(\psi)$ vanishes.*

Now the function $g(u)$ defined by means of the series $\sum\limits_0^\infty \frac{c_n}{u^{n+1}}$ which is convergent if $|u| > k$, may be easily proved, by means of the formula,

$$g(u) = \sum_0^\infty \frac{c_n}{u^{n+1}} = \int_0^{\infty(\psi)} \phi(z)\, e^{-uz}\, dz, \qquad \ldots \quad (2.3)$$

to be an analytic function of u in the region lying exterior to the envelope Σ of the lines

$$R\cos(\theta + \psi) = \lambda(\psi), \qquad (|\psi| \leqslant \pi), \qquad \ldots \quad (2.4)$$

where $$u = Re^{i\theta}.$$

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* All these properties of $\lambda(\psi)$ follow from the classical memoir of Phragmén and Lindelöf, 'Sur une extension d'un principe classique de l'analyse, etc.,' *Acta Mathematica*, Vol. 31, 1908, pp. 381—406.

THE JOURNAL

OF

THE INDIAN MATHEMATICAL SOCIETY

NOTES AND QUESTIONS.

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Volume XVIII, No. 10.

August 1930.

Notes and Questions.

New Light on Bhaskara's Chakravala or Cyclic Method of solving Indeterminate Equations of the Second Degree in two Variables.

1. Introduction.

The equation $x^2 - Ny^2 = 1$* has a long and interesting history† behind it of more than 2000 years. The earliest civilized nations of the world, the Indians and the Greeks, were fascinated by the problem. While the latter never gave more than particular solutions for some simple cases like $2x^2 - y^2 = \pm 1$, the former were the first to realize the true inwardness of the problem and to give a general solution based on the principle of composition of quadratic forms. More than thousand years before Euler re-discovered, after spending much thought and labour on the problem, what he calls *the following remarkable theorem* which contains within it the foundation of higher solutions:

" If $x = a,\ y = b$ satisfies $\alpha x^2 + p = y^2$,

and $x = c,\ y = d$ satisfies $\alpha x^2 + q = y^2$,

then $x = bc \pm ad,\ y = bd \pm \alpha ac$ satisfies $\alpha x^2 + pq = y^2$ "

we find Brahmagupta dealing with the same theorem in a special chapter 'Vargaprakriti' and drawing important corollaries from it. In particular, he derives the identity

$$\alpha\left(\frac{2\,ab}{p}\right)^2 + 1 = \left(\frac{b^2 + \alpha a^2}{p}\right)^2 \text{ from } \alpha a^2 + p = b^2.$$

Special identities are also given by him for deriving integral solutions of $Ny^2 + 1 = x^2$ from those of $Ny^2 \pm 4 = x^2$.

In the period between Brahmagupta (born 598 A.D.) and Bhaskara

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* It is understood throughout this paper that N is a non-square integer.

† There is a whole book '*Geschichte der Gleichung* $t^2 - Du^2 = 1$' devoted to this history by H. Konen. It contains a comparison between Lagrange's procedure and the Indian. The book, however, has not been accessible to me for direct reference.

155326

(born 1114 A.D.), there must have taken place some modifications* on Brahmagupta's work, of which, however, we have little knowledge, except from the indirect references thrown out by Bhaskara. About 1150 A.D., Bhaskara surprises us with his brilliant method for integral solutions, designating it as 'Chakravala' after the ancients. Thus, indirectly Bhaskara throws the hint that he does not claim it as his own. But it is certainly not Brahmagupta's. Owing probably to the fact that the mathematical writings of Brahmagupta and Bhaskara, separated though they are by about five centuries, are put together in the same volume by H. T. Colebrooke, the credit that is really Bhaskara's is sometimes shared by Brahmagupta also. Referring to the cyclic method, T. L. Heath says (on p. 281 of his *Diophantus*): "If the Greeks did not accomplish the general solution of our equation, it is all the more extraordinary we should have such a general solution in practical use among the Indians as early as the time of Brahmagupta under the name of the 'Cyclic method'; ... ...

... ...

"The 'cyclic method' of solving the equation $x^2 - Ay^2 = 1$ is found in Brahmagupta and Bhaskara and is well described by Hankel, Cantor, and Konen."

Thus, the cyclic method is wrongly attributed to Brahmagupta also. In the absence of explicit reference to this method in the works of the earlier writers, it is fair to take it as Bhaskara's; for there is an element of doubt in the interpretation of 'चक्रवालमिदं जगुः' and Bhaskara may be after all giving credit to his predecessors merely for the name 'चक्रवाल' as applied to all iterative operations in mathematics.

The true nature of the Chakravala has not been understood by many eminent authorities from the time of Colebrooke onwards. This accounts for its neglect even by Colebrooke who speaks so highly of one of Bhaskara's rational solutions which happens to be exactly the same which Lord Brouncker devised to answer a question proposed by way of

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*As an evidence of such modifications may be noted the fact that the technical terms 'आदि' and 'अन्त्य' used by Brahmagupta for the two variables (y, x) in the equation $Ny^2 + 1 = x^2$ have changed by Bhaskara's time to 'ह्रस्व' and 'ज्येष्ठ.'

challenge by Fermat in 1657. H. J. Smith in his reports* on the Theory of Numbers, remarks, probably on the authority of Colebooke, that Lord Brouncker and Wallis first gave rational solutions just like Bhaskara and Brahmagupta 700 years before them, having misunderstood the nature of the problem. Surely Bhaskara and Brahmagupta did not misunderstand the problem, like Lord Brouncker and Wallis. The Indians never proceeded along the superficial lines adopted by Diophantus in his Lemma to Book VI, 15 (*vide* Heath's *Diophantus*, p. 238) but showed their grip over the problem by enunciating the important formula, which is equivalent to tho composition of quadratic forms rediscovered by Euler, a thousand years later. Again, T. L. Heath himself, on p. 285 of his *Diophantus*, holds that the Indian method is the same as that rediscovered and expounded by Lagrange in his memoir of 1768. He is probably inclined to this view through H. Konen's comparison of the two methods—the Indian and the Lagrangean—in his book on the history of the equation.

One may also mention in this connection the definitely contemptuous view of the late G. R. Kaye in several of his articles. In the article on 'The Source of Hindu Mathematics' (J. R. A. S. 1910), he writes:

" Indeterminate equations play an important part in Hindu mathematics and the discovery of solutions of $Du^2 + 1 = t^2$ in Hindu works of a fairly early date was considered very remarkable. Possibly the fact that Fermat, Wallis, Brouncker, Euler, Lagrange, and others paid considerable attention to the problem, gave the discovery of it in Hindu works *a somewhat fictitious value*. ... ...

Bhaskara gives *some* alternative methods for the solution of the Pellian Equation but *in no essential does he improve on Brahmagupta*."†

Again, on p. 672, *East and West*, (Vol. XVII, No. 201, July 1918) we have such statements as:

"The discovery of Hindu works on rational, integral solutions of the so-called Pellian Equation $ax^2 + 1 = y^2$ made something of *a splash*.

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* *Vide Collected Mathematical Papers* of H. J Smith (Edited by J W. L. Glaisher) Vol. I pp 192—200.

† It will be easily conceded after reading this paper how Bhaskara's Cyclic Method is really an advance on Brahmagupta's.

The very problem had not long before been the subject of correspondence and discussion in Europe and this gave emphasis to the discovery of the Hindu solution and the combination of events perhaps led to an over-estimate of the value of the Hindu contribution. More recent research leads to the conclusion that the very occurrence of these Hindu solutions is a result of Greek influence."

Lastly, it is unfortunate that even K. J. Sanjana (in the J. I. M. S. Notes and Questions, Vol. XVIII, No. 1, p. 22), refers to Bhaskara's method of solving the equation $x^2 - 67y^2 = 1$ as a process of converting $\sqrt{67}$ into a continued fraction and gives a table* of his own to show the reader that Bhaskara's process tallies exactly (this, of course, is not a fact) with that outlined by Chrystal.

It is the object of this paper to bring out at some length the true inwardness of Bhaskara's cyclic method and to point out that the Chakravala is not the same as either the continued fraction method or the crude method of Brouncker and Wallis. On the other hand, it will be shown in the sequel that Bhaskara's method is more in line with Gauss's solution based on his theory of Quadratic Forms. The present investigation will reveal that Bhaskara's method leads naturally to a new set of reduced forms distinct from those of Gauss, Lagrange, Klein, and Hermite. Further, Bhaskara's method does not lead straight to a simple continued fraction but to an irregular one. It has also the advantage of reducing the number of recurring elements and thus may claim a certain superiority to the other methods in the matter of practical computation.

2. The Chakravala or Cyclic Method explained.

Brahmagupta's formula for the composition of quadratic forms, *viz.*:

$$(XX' \pm \alpha YY')^2 - \alpha(XY' \pm X'Y)^2 = (X^2 - \alpha Y^2)(X'^2 - \alpha Y'^2)$$

is also repeated by Bhaskara. Both of them realise that with the help of this, the solutions of the equation

$$Ny^2 + 1 = x^2$$

* Mr. Sanjana wrongly believes that his table represents Bhaskara's method. Bhaskara's solution agrees with Sanjana's table only as far as the first three steps and from the third step Bhaskara proceeds immediately to Sanjana's fifth step and from thence takes a short cut to the required solution.

can be derived easily from those of

$$Ny^2 \pm 2 = x^2, \text{ or } Ny^2 \pm 4 = x^2.$$

Bhaskara has attempted to go one step further to show that the roots of the equations

$$Ny^2 + 1 = x^2, \; Ny^2 \pm 2 = x^2, \; Ny^2 \pm 4 = x^2$$

can be derived, by successive reduction, from those of the more general equation

$$Ny^2 \pm k = x^2 *$$

where k is any integer. The method of such reduction is outlined as the *Chakravala*:

ह्रस्वज्येष्ठपदक्षेपान्भाज्यप्रक्षेपभाजकान् ॥

कृत्वा कल्प्यो गुणस्तत्र तथा प्रकृतितश्च्युते ।

गुणवर्गे प्रकृत्योनेऽथवाल्पं शेषकं यथा ॥

तत्तु क्षेपहृतं क्षेपः व्यस्तः प्रकृतितश्च्युते ।

गुणलब्धिः पदं ह्रस्वं ततोज्येष्ठमतोऽसकृत् ॥

त्यक्त्वा पूर्वपदक्षेपांश्चक्रवालमिदं जगुः ।

चतुर्द्व्येकयुतावेवमभिन्ने भवतः पदे ॥

चतुर्द्विक्षेपमूलाभ्यां रूपक्षेपार्थभावना ॥ †

Considering the smaller root, the greater root, and the additive (of the given equation) as respectively the dividend, addend, and divisor (of a linear indeterminate equation), the multiplier should be so determined that the square of this multiplier being subtracted from the given co-efficient or the co-efficient being subtracted from the square of the multiplier (as the case may be), the residue is the least (under the circumstances); this residue divided by the original additive is the (next)

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* In the Indian rule, N is called the co-efficient, y the lesser root, and x the greater root while ($\pm k$) is known as the addend or additive. The terms 'lesser' and 'greater' are not to be taken in their literal sense, just as the word 'imaginary' in modern mathematics should not be understood too literally.

† *Vide* p. 185, verses 46 – 50; *Bijaganita* of Bhaskaracharya with the Commentary of Durga Prasad Trivedi, 2nd Edn., Lucknow, 1917, hereafter referred to as Bijaganita.

additive, being reversed in sign if the residue be obtained by subtraction from the co-efficient. The quotient corresponding to the multiplier in the linear equation is the smaller root, whence the greater root can be obtained. This process may be repeated with the new additive, smaller and greater roots, in place of the old ones. Such an iterative process is styled *Chakravala*. By this method, we get integral roots for equations with additives ± 4, ± 2, ± 1, whence may be derived by the principle of composition, the roots of the equation with *unity* as the additive.

For a clearer understanding of the above rule, one has only to restate it in modern terms thus:

Let the roots of the equation

$$Ny^2 + k = x^2 \qquad \ldots \quad (1)$$

be $x = a,\ y = b$ (N being a positive, non-square integer).

Solve the linear indeterminate equation

$$\frac{bx + a}{k} = y$$

in integers and choose a value l for x so that $| l^2 - N |$ may be the least possible.

Then, the corresponding value for y is $\frac{bl + a}{k}$.

Bhaskara says that the integral roots of a new equation

$$Ny^2 + \frac{l^2 - N}{k} = x^2 \qquad \ldots \quad (2)$$

are given by $$y = \frac{bl + a}{k}$$

and hence $$x = \frac{al + Nb}{k}$$

where $(l^2 - N)/k$ is also an integer.

Just as we derived (2) from (1), we can derive another equation from (2) with known roots and so on, until the additive ultimately reduces to * ± 4, ± 2, or ± 1.

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* Bhaskara's own worked examples illustrate the cases where the additive reduces to -4, and -2; thus, starting with the known solution ($x = 8$, $y = 1$

As T. L Heath says, nothing is wanting to the cyclic method except the proof that it will in every case lead to the desired result, whenever N is a non-square integer; but he is wrong in supposing that Lagrange was the first to supply the proof for Bhaskara's method. Indeed, the first complete proof of Bhaskara's Cyclic Method is the one sketched in this paper.

3. The Theory underlying the Cyclic Method.

Lemma. If a, b, a_1, b_1, be integers such that $ab_1 - a_1b = 1$, then the equations

$$x^2 - Ny^2 = k \quad \ldots \quad (1)$$

and

$$x'^2(a^2 - Nb^2) + 2x'y'(aa_1 - Nbb_1) + y'^2(a_1^2 - Nb_1^2) = k \ \ldots \quad (2)$$

are so related that their integral solutions can be put into one to-one correspondence by the linear relation

$$x = ax' + a_1y'$$
$$y = bx' + b_1y'.$$

For, it is readily seen that (2) is derived from (1) by the linear substitution; solving for x', y' in terms of x, y, we get

$$x' = \frac{b_1x - a_1y}{ab_1 - a_1b} = b_1x - a_1y$$

$$y' = \frac{bx - ay}{ba_1 - ab_1} = -bx + ay.$$

This shows that, if (x', y') are integral, so are (x, y) and *vice-versa*. Thus any solution of one of the equations gives simultaneously a solution of the other.

In the language of the Theory of Numbers, the two quadratic forms on the left-side of (1) and (2) are said to be equivalent.

Now, let (a, b) be an integral solution of (1), so that

$$a^2 - Nb^2 = k.$$

Put $\quad aa_1 - Nbb_1 = l$ and $a_1^2 - Nb_1^2 = k_1$.

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of the equation $61y^2+3=x^2$ he gets at the next stage of reduction the solution $(x=39, y=5)$ of the equation $61y^2-4=x^2$, where the additive is -4; similarly, one solution of the equation $67y^2-3=x^2$ leads successively to the solutions of the equations $67y^2+6=x^2$, $67y^2-7=x^2$, and $67y^2-2=x^2$, (the last of these equations having the additive -2).

It is readily verified that

$$\left.\begin{aligned} l^2 - kk_1 &= \mathrm{N}\,(ab_1 - a_1b)^2 = \mathrm{N}. \\ \textit{i.e.,} \quad a_1{}^2 - \mathrm{N}b_1{}^2 &= k_1 = \frac{l^2 - \mathrm{N}}{k} \end{aligned}\right\} \quad \ldots \quad (3)$$

Also, solving the two equations

$$ab_1 - a_1b = 1$$

and

$$aa_1 - \mathrm{N}bb_1 = l \qquad \text{for } a_1,\ b_1$$

we get

$$\left.\begin{aligned} a_1 &= \frac{al + \mathrm{N}b}{k} \\ b_1 &= \frac{bl + a}{k} \end{aligned}\right\} \quad \ldots \quad (4)$$

The results (3) and (4) lay bare the clue to Bhaskara's method. In fact, they suggest a method of deriving a pair of integers $(a_1.\ b_1)$ and a triplet $(k,\ l.\ k_1)$ from a solution $(a,\ b)$ of equation (1), by taking

$$b_1 = \frac{bl + a}{k},\ k_1 = \frac{l^2 - \mathrm{N}}{k},\ \text{and } \mathrm{N}b_1{}^2 + k_1 = a_1{}^2,$$

where all the letters represent integers.

Thus, if $(a,\ b)$ be a pair of integral values satisfying the equation

$$\mathrm{N}y^2 + k = x^2$$

then $(a_1,\ b_1)$ are also a pair of integers satisfying a new equation

$$\mathrm{N}y^2 + k_1 = x^2 \qquad \ldots \quad (5)$$

Treating the roots $(a_1,\ b_1)$ of (5) in the same way as we treated those of (1), we get the pair $(a_2,\ b_2)$ and the triplet $(k_1,\ l_1,\ k_2)$ such that $(a_2,\ b_2)$ satisfy the equation $\mathrm{N}y^2 + k_2 = x^2$ and so on.

In the notation of the Theory of Numbers, the forms $(1,\ 0,\ -\mathrm{N})$ $(k,\ l,\ k_1)$, $(k_1,\ l_1,\ k_2)$, ... obtained by the above method are all equivalent.

Bhaskara's proposition is that the k's ultimately reach the values ± 4, ± 2, or ± 1.

We now proceed to show that the k's reach 1 always, but they need not necessarily pass through the values $\pm 2, \pm 4, -1$ before reaching 1.

If they do, however, take these values, Bhaskara's idea is to take a short-cut from there to the required solution. (*Vide*, Bhaskara's own solution to the equations $61y^2 + 1 = x^2$ and $67y^2 + 1 = x^2$, pp. 194—198, *Bijaganita*).

Recurrence Formulae.

The relations between the successive pairs (a, b), (a_1, b_1), ... as well as those between the successive triplets (k, l, k_1), (k_1, l_1, k_2), ... are as follows:

$$\left.\begin{array}{l} a^2 - Nb^2 = k; \\ Nb_1^2 + k_1 = a_1^2,\ a_1 = \dfrac{al + Nb}{k},\quad b_1 = \dfrac{bl + a}{k},\quad k_1 = \dfrac{l^2 - N}{k}; \\ Nb_2^2 + k_2 = a_2^2,\ a_2 = \dfrac{a_1 l_1 + Nb_1}{k_1},\ b_2 = \dfrac{b_1 l_1 + a_1}{k_1},\ k_2 = \dfrac{l_1^2 - N}{k_1}; \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ Nb_r^2 + k_r = a_r^2,\ a_r = \dfrac{a_{r-1} l_{r-1} + Nb_{r-1}}{k_{r-1}},\ b_r = \dfrac{b_{r-1} l_{r-1} + a_{r-1}}{k_{r-1}}, \\ \qquad\qquad\qquad\qquad k_r = \dfrac{l_{r-1}^2 - N}{k_{r-1}}. \end{array}\right\} \quad (6)$$

$$\left.\begin{array}{l} \text{Now,}\quad a_2 = \dfrac{a_1 l_1 + Nb_1}{k_1} = \dfrac{a_1 l_1}{k_1} + \dfrac{Na}{kk_1} + \dfrac{l}{k_1}\left(a_1 - \dfrac{al}{k}\right) \\ \qquad\qquad = a_1 . \dfrac{l + l_1}{k_1} + \dfrac{a(N - l^2)}{kk_1} \\ \qquad\qquad = a_1 . \dfrac{l + l_1}{k_1} - a \\ \text{Similarly, } b_2 = b_1 . \dfrac{l + l_1}{k_1} - b \end{array}\right\} \ldots \quad (7)$$

Since $ab_1 - ba_1 = a_1 b_2 - b_1 a_2 = \ldots\ldots = 1$, and we may take a, b to be prime to each other and have the same sign, it follows that the pairs (a_1, b_1), (a_2, b_2), ... also contain relatively prime integers having the same sign.

If $\dfrac{l + l_1}{k_1}$ be not an integer, then k_1 or some one of its factors should

30

go into both a_1 and b_1, which is contrary to the fact that a_1 and b_1 are relatively prime. Hence, $\frac{l + l_1}{k_1}$ must be an integer.

We may write (7) in the more general form as giving the relation between any three successive pairs (a_r, b_r), (a_{r+1}, b_{r+1}), and (a_{r+2}, b_{r+2}) thus:

$$\left.\begin{aligned} a_{r+2} &= a_{r+1} \cdot \frac{l_r + l_{r+1}}{k_{r+1}} - a_r \\ b_{r+2} &= b_{r+1} \cdot \frac{l_r + l_{r+1}}{k_{r+1}} - b_r \end{aligned}\right\} \quad \ldots \quad (8)$$

where $(l_r + l_{r+1}) / k_{r+1}$ is an integer.

Incidentally, we may also notice the relation

$$a_r - b_r l_{r-1} = - b_{r-1} k_r. \quad \ldots \quad (9)$$

In the usual notation of the Theory of Numbers. the triplets

$$(k, l, k_1) \; (k_1, l_1, k_2), \; \ldots \; \ldots$$

represent *equivalent adjacent* quadratic forms with determinant N, since

$$l^2 - kk_1 = l_1^2 - k_1 k_2 = \ldots \ldots = \text{N}.$$

and $\quad l + l_1 \equiv 0 \pmod{k_1}, \;\; l_1 + l_2 \equiv 0 \pmod{k_2}, \;\ldots$

It is important to note that these latter congruences can easily take the place of Bhaskara's indeterminate equations

$$\frac{b_1 x + a_1}{k_1} = y, \; \frac{b_2 x + a_2}{k_2} = y, \; \ldots \ldots$$

Bhaskara's Condition for Reduction.

We will now make use of the important condition 'अल्पं शेषकं यथा' in Bhaskara's method of reduction which distinguishes it from the reduced forms due to Gauss, Lagrange and others.

Bhaskara's condition is that l_r should be such a value of x in the indeterminate equation

$$\frac{b_r x + a_r}{k_r} = y$$

that $| l_r^2 - \text{N} |$ is the least. $\quad \ldots \quad$ (10)

The values of x satisfying (10) are, by virtue of (9) above, of the form $-l_{r-1} + n \mid k_r \mid$ where n can take all integral values, positive, zero, and negative. In this singly infinite system of values, there are bound to be a set of two integers (one of which is less and the other greater than $\sqrt{N}$) in the immediate neighbourhood of $+\sqrt{N}$ and similarly another in the immediate vicinity of $-\sqrt{N}$. The squares of these four *integers are evidently nearer to N than the square of any other value of x in the equation (10). According to Bhaskara we have to choose the nearest of these four squares to N.† An exceptional case may occur when the root corresponding to the nearest square leads back to the previous step in the process of reduction. In this case, the root corresponding to the nearest of the remaining squares should be chosen. This exceptional case is not explicitly noted by Bhaskara though it may be implied in his cryptic expression ‡

We will first consider the case when $\mid k_r \mid > \sqrt{N}$.

Let the least positive value of x in (10) be L.

Then $L < \mid k_r \mid$ and the four squares in question which are nearer to N than all the others are

$$L^2,\ (L + \mid k_r \mid)^2,\ (L - \mid k_r \mid)^2 \text{ and } (L - 2 \mid k_r \mid)^2$$

of which,

$$(L + \mid k_r \mid)^2 \text{ and } (L - 2 \mid k_r \mid)^2$$

are both greater than k_r^2 and therefore greater than N, and it can be verified that the nearer of these two squares is always further from N than the nearer of the other two squares, *viz.*, L^2, and $(L - \mid k_r \mid)^2$ whether the latter be both greater than N or less than N or only one of them is less than N.

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* Sometimes either three or two due, to the co-incidence of some of the integers.

† This point has been completely overlooked by writers on Bhaskara. It is interesting to note that, for Gauss's reduction, it is sufficient if we take that value of x in (10) which is nearest to $\sqrt{N}$ and less than $\sqrt{N}$. Thus Bhaskara's condition goes a step further than Gauss's and lacks thereby a certain amount of simplicity.

‡ Bhaskara seems to recognise this necessary modification as for example in his own solution of the equation $61y^2 + 1 = x^2$.

For let $(L + |k_r|)^2$ be nearer to N than $(L - 2|k_r|)^2$: and so $|k_r| - L > L$. If $(L - |k_r|)^2$ is nearer to N than L^2, we have obviously $(L + |k_r|)^2 - N > N - (L - |k_r|)^2$ when $(L - |k_r|)^2 < N$, and $(L + |k_r|)^2 - N > (L - |k_r|)^2 - N$ when $(L - |k_r|)^2 > N$.

If, however, L^2 be nearer to N than $(L - |k_r|)^2$, then, evidently $(L + |k_r|)^2 - N > L^2 - N$ when $L > \sqrt{N}$. When $L < \sqrt{N}$, $|k_r| - L$ cannot also be less than $\sqrt{N}$, for this implies

$$\sqrt{N} > |k_r| - L > L$$

which contradicts the assumption that

$$L^2 \text{ is nearer to N than } (L - |k_r|)^2.$$

Hence, $$|k_r| - L > \sqrt{N} > L$$

$$\therefore \quad (L + |k_r|)^2 - N > (|k_r| - L)^2 - N > N - L^2.$$

A similar argument applies to the case where $(L - 2|k_r|)^2$ is nearer to N than $(L + |k_r|)^2$, to prove that $(L - 2|k_r|)^2 - N$ is greater than the lesser of the two differences

$$N \sim L^2 \text{ and } N \sim (L - |k_r|)^2.$$

From the above discussion, it becomes evident that the value l_r satisfying Bhaskara's condition is either

$$L \text{ or } L - |k_r| \quad \text{when} \quad |k_r| > \sqrt{N}.$$

Hence $$|l_r| < |k_r|,$$

Further, since $$l_r^2 - k_r k_{r+1} = N,$$

we have $$|k_r| \cdot |k_{r+1}| < l_r^2 \text{ or } N$$

and in either case $$|k_{r+1}| < |k_r|$$

Therefore, as long as the k's are greater than $\sqrt{N}$. Bhaskara's method of reduction leads to smaller and smaller numerical values for k, so that at some stage

$$|k_r| < \sqrt{N}, \text{ while } |k| > |k_1| > |k_2| > \dots > k_{r-1} > \sqrt{N}.$$

Let us now consider the case $|k_r| < \sqrt{N}$. Since $|l_r^2 - N|$ is least, we must have

$$|(l_r \pm k_r)^2 - N| > |l_r^2 - N| \qquad \ldots \quad (11)$$

But, here we take into account only the squares of the number-pairs nearest to $\sqrt{N}$ or $-\sqrt{N}$ and not the nearest of all the four squares as contemplated on p. 235.*

The inequality relation (11) can be transformed as follows :

(i) When $l_r^2 > N$, $|l_r| - |k_r| < \sqrt{N}$ and so

$$N - (|l_r| - |k_r|)^2 > l_r^2 - N$$

i.e., $$N - l_r^2 + 2|l_r|\,|k_r| - k_r^2 > l_r^2 - N$$

i.e., $$|k_r|(2|l_r| - |k_r|) > 2(l_r^2 - N)$$
$$> 2|k_r|\,|k_{r+1}|$$

$\therefore$ $$|l_r| > |k_{r+1}| + \tfrac{1}{2}|k_r| \qquad \ldots \quad (12)$$

Squaring both sides,

$$l_r^2 > k_{r+1}^2 + \tfrac{1}{4}k_r^2 + |k_r|\,|k_{r+1}|$$

$\therefore$ $$k_{r+1}^2 + \tfrac{1}{4}k_r^2 < N\,\dagger \qquad \ldots \quad (13)$$

(ii) When $l_r^2 < N$, $|l_r| + |k_r| > \sqrt{N}$ and so

$$(|l_r| + |k_r|)^2 - N > N - l_r^2$$

from which we easily derive as in (i)

$$|l_r| > |k_{r+1}| - \tfrac{1}{2}|k_r| \qquad \ldots \quad (14)$$

which, on squaring, leads again to (13).

---

* In this connection, it may be mentioned, that if p, q be positive integers nearest to $\sqrt{N}$ and p', q' the negative integers nearest to $-\sqrt{N}$, where p, q, p', q' are the values of x satisfying the condition (10), then the two squares nearest to N can be only one of the sets :

$$(p^2, p'^2), (p^2, q'^2), (q^2, p'^2), (q^2, q'^2).$$

Suppose, for example, (p^2, p'^2) are the nearest squares to N and we have to reject one of them say p^2, as it leads back to the previous step in Bhaskara's method of reduction. Then the next square we have to choose is either p'^2 or q'^2 which corresponds to the number-pair nearest to $-\sqrt{N}$, and thus even in the exceptional case the relation (11) holds.

† Bhaskara's condition of reduction (10) amounts to more than (13). It implies further that $|k_{r+1}|$ should be the least of the possible values with the exception of $|k_{r-1}|$.

Thus the condition $k_{r+1}^2 + \frac{k_r^2}{4} < N$ is equivalent to

(i) $(l_r \pm k_r)^2 \sim N > l_r^2 \sim N$

(ii) $| l_r | > \frac{1}{2} | k_r | + | k_{r+1} |$ when $l_r^2 > N$

and (iii) $| l_r | > | k_{r+1} | - \frac{1}{2} | k_r |$ when $l_r^2 < N$,

Similarly, the condition $k_r^2 + \frac{k_{r+1}^2}{4} < N$ can be seen to be equivalent to (i) $(l_r \pm k_{r+1})^2 \sim N > l_r^2 \sim N$,

and (ii) $| l_r | > \frac{1}{2} | k_{r+1} | + | k_r |$, or

$| l_r | > | k_r | - \frac{1}{2} | k_{r+1} |$

according as l_r^2 is greater or less than N.

Cor. (1) In every case, when $| k_r | < \sqrt{N}$, $| k_{r+1} | < \sqrt{N}$.

Cor. (2) Since $l_r^2 - k_r k_{r+1} = N$, and $| k_r \cdot k_{r+1} | < N$.

$l_r < \sqrt{2N}$.

Cor. (3) $| l_r | \sim \sqrt{N} = \frac{| k_r | \, | k_{r+1} |}{| l_r | + \sqrt{N}} < | k_r |$ and $| k_{r+1} |$.

Cor. (4) Since $| k_r |$, $| l_r |$, $| k_{r+1} |$ can take only a limited set of integral values, the number of forms of the type (k_r, l_r, k_{r+1}) is necessarily finite.

Reduced Forms.

When the k's have become numerically less than $\sqrt{N}$, since they cannot continuously decrease numerically, we must have at some stage

$$| k_{r+1} | \geqslant | k_r | .$$

The corresponding reduced form evidently satisfies two conditions :

$$\text{(i)} \quad k_r^2 + \frac{k_{r+1}^2}{4} < N$$

$$\text{(ii)} \quad k_{r+1}^2 + \frac{k_r^2}{4} < N$$

We shall call a form (A, B, C) of determinant N. a *reduced Bhaskara form* (or simply a *Bhaskara form*), if it satisfies the conditions

$$A^2 + \frac{C^2}{4} < N, \quad C^2 + \frac{A^2}{4} < N.$$

In the method of reduction advocated by Bhaskara, the existence of such reduced forms as (A, B, C) has now been proved.

Incidentally, we prove also the following

THEOREM. A reduced Bhaskara form can always be found equivalent to any proposed form.

Properties of Bhaskara Forms.

Let (k_r, l_r, k_{r+1}) be a Bhaskara form and $(k_{r+1}, l_{r+1}, k_{r+2})$ be an adjacent form obtained from the former by Bhaskara's method.

Since $(l_r \pm k_{r+1})^2 \sim N > l_r^2 \sim N$, (*vide* p. 238), $-l_r$ is a possible value of l_{r+1}; but this has to be rejected as it leads back to k_r, and so we have to choose for l_{r+1} one of the two numbers nearest to $+\sqrt{N}$ or $-\sqrt{N}$ according as $-l_r$ belongs to the number pair nearest to $-\sqrt{N}$ or $+\sqrt{N}$.

This shows that l_{r+1} and l_r are of the same sign. Further, it is easily seen, from the relative magnitudes of the two squares nearest to N, that

$$l_{r+1} \sim l_r < | k_{r+1} |.$$

We may suppose, for convenience, that l_r is positive. The proofs applicable to this case apply also to the case where l_r is negative, for we may write $| l_r |$ for l_r.

With these preliminaries settled, we can now prove the theorem.

THEOREM I. *A successor of a Bhaskara form is also a Bhaskara form.*

If $(k_{r+1}, l_{r+1}, k_{r+2})$ be a successor to the reduced form (k_r, l_r, k_{r+1}) then it is evident that the successor is also a reduced form in the

following cases :

(i) $|k_{r+1}| \leqslant |k_{r+2}|$.

(ii) $|k_{r+2}| \leqslant |k_r|$.

(iii) $l_{r+1} > \sqrt{N}, l_{r+1} \geqslant \frac{3}{2} |k_{r+1}|$, and $|k_{r+1}| > |k_{r+2}| > |k_r|$.

(iv) $l_{r+1} < \sqrt{N}, l_{r+1} > |k_{r+1}|$, and $|k_{r+1}| > |k_{r+2}| > |k_r|$.

So, the theorem remains to be proved for the remaining cases :

(v) $l_{r+1} > \sqrt{N}, l_{r+1} < \frac{3}{2} |k_{r+1}|$, and $|k_{r+1}| > |k_{r+2}| > |k_r|$

and (vi) $l_{r+1} < \sqrt{N}, l_{r+1} < |k_{r+1}|$, and $|k_{r+1}| > |k_{r+2}| > |k_r|$

Case (v). Since $l_{r+1}^2 - N = |k_{r+1} \cdot k_{r+2}| > |k_r| \, |k_{r+1}|$

and $$|l_r^2 - N| = |k_r k_{r+1}|,$$

we have $$l_{r+1} > l_r, \text{ whether } l_r \gtrless \sqrt{N}$$

$\therefore$ $$l_r + l_{r+1} < 2 l_{r+1} < 3 |k_{r+1}|$$

Also, $$l_{r+1} > \sqrt{N} > |k_{r+1}|$$

and $$l_r + l_{r+1} \equiv 0 \pmod{|k_{r+1}|}$$

Hence, $$l_r + l_{r+1} = 2|k_{r+1}|$$

But $$l_{r+1} - l_r < |k_{r+1}|$$

i.e., $$2|k_{r+1}| - 2 l_r < |k_{r+1}|$$

which implies that $$l_r < |k_{r+1}| < \sqrt{N}.$$

Now, $$l_{r+1}^2 - N > N - l_r^2$$

i.e., $$\{2|k_{r+1}| - l_r\}^2 + l_r^2 > 2N$$

i.e., $$l_r^2 + 2 k_{r+1}^2 - 2 l_r |k_{r+1}| > N$$

i.e., $$2 k_{r+1}^2 - 2 l_r |k_{r+1}| > N - l_r^2 > |k_r k_{r+1}|$$

i.e., $$2|k_{r+1}| - 2 l_r > |k_r|$$

i.e., $$l_r < |k_{r+1}| - \tfrac{1}{2}|k_r|$$

which contradicts the condition

$$l_r > |k_{r+1}| - \tfrac{1}{2}|k_r|$$

equivalent to $$k_{r+1}^2 + \tfrac{1}{4} k_r^2 < N,$$

This investigation reveals that *Case* (v) is an impossible one.

Case (vi). As before, $| l_{r+1}^2 - N | > | l_r^2 - N |$.

But, since $l_{r+1} < \sqrt{N}$,

we get $N - l_{r+1}^2 > N - l_r^2$ or $l_r^2 - N$

and in either case $l_r > l_{r+1}$.

We know that $l_r - l_{r+1} < | k_{r+1} |$

$\therefore$ $l_r < l_{r+1} + | k_{r+1} |$

$< 2 | k_{r+1} |$

so that, $l_r + l_{r+1} < 3 | k_{r+1} |$.

Since $l_r + l_{r+1} \equiv 0 \pmod{| k_{r+1} |}$

we have $l_r + l_{r+1} = | k_{r+1} |$ or $2 | k_{r+1} |$.

Again, $(l_{r+1})^2 + | k_{r+1} | . | k_{r+2} | = N = l_r^2 \pm | k_r | . | k_{r+1} |$

$\therefore$ $l_r^2 - l_{r+1}^2 = | k_{r+1} | \{ | k_{r+2} | \mp | k_r | \}$

If $l_r + l_{r+1} = | k_{r+1} |$, then $l_r - l_{r+1} = | k_{r+2} | \mp | k_r |$.

so that $l_r = \frac{1}{2} | k_{r+1} | + \frac{1}{2} | k_{r+2} | \mp \frac{1}{2} | k_r | < | k_{r+1} | \mp \frac{1}{2} | k_r |$

the upper or lower sign to be taken in each case according as l_r is less or greater than $\sqrt{N}$. This contradicts the condition that (k_r, l_r, k_{r+1}) is a reduced form (*vide* p. *238*). Hence $l_r + l_{r+1} \neq | k_{r+1} |$.

Next, if $l_r + l_{r+1} = 2 | k_{r+1} |$, then $l_r - l_{r+1} = \frac{1}{2} | k_{r+2} | \mp \frac{1}{2} | k_r |$,

so that $l_{r+1} = | k_{r+1} | - \frac{1}{4} \{ | k_{r+2} | \mp | k_r | \}$

$> | k_{r+1} | - \frac{1}{2} | k_{r+2} |$

which is equivalent to the condition $k_{r+1}^2 + k_{r+2}^2 / 4 < N$.

Thus in all cases, we prove that the form $(k_{r+1}, l_{r+1}, k_{r+2})$ satisfies the conditions

$$k_{r+2}^2 + k_{r+1}^2 / 4 < N$$

and $$k_{r+1}^2 + k_{r+2}^2 / 4 < N.$$

Hence, the theorem is proved.

THEOREM II. *Two different Bhaskara forms cannot have the same successor.*

If possible, let (k_r, l_r, k_{r+1}) and $(k_{r+1}, l_{r+1}, k_{r+2})$ be two consecutive Bhaskara forms, as also (k', l', k_{r+1}) and $(k_{r+1}, l_{r+1}, k_{r+2})$.

Then
$$l_r + l_{r+1} \equiv 0 \ (\text{mod.} \mid k_{r+1} \mid)$$
$$l' + l_{r+1} \equiv 0 \ (\quad ,, \quad)$$
$$\therefore \quad l' \sim l_r \equiv 0 \ (\quad ,, \quad) \qquad \ldots \quad (1)$$

But, since* $\mid l_r \sim \sqrt{N} \mid < \mid k_{r+1} \mid$ and $\mid l' \sim \sqrt{N} \mid < \mid k_{r+1} \mid$,
$$\therefore \quad \mid l' - l_r \mid < \mid l_r - \sqrt{N} \mid + \mid l' - \sqrt{N} \mid < 2 \mid k_{r+1} \mid \ \ldots \quad (2)$$

From (1) and (2), we infer
$$\mid l' - l_r \mid = 0 \text{ or } \mid k_{r+1} \mid$$

From the condition of reduction,
$$\mid (l' \pm k_{r+1})^2 - N \mid > \mid l'^2 - N \mid$$
which reduces to
$$\mid l^2_r - N \mid > \mid l'^2 - N \mid$$
where
$$\mid l' - l_r \mid = \mid k_{r+1} \mid$$
A similar argument proves also the contradictory result
$$\mid l'^2 - N \mid > \mid l_r^2 - N \mid$$
$$\therefore \quad \mid l' - l_r \mid \neq \mid k_{r+1} \mid.$$

Hence $\mid l' - l_r \mid = 0$, *i.e.* $l' = l_r$, from which it follows that $k' = k_r$; *i.e.* the two forms (k_r, l_r, k_{r+1}) and (k', l', k_{r+1}) are identical.

Thus, the theorem is proved.

THEOREM III. *The Bhaskara forms repeat in a cycle and the first member of the cycle is the same as the first Bhaskara form obtained in the course of reduction.*

For, if otherwise, two different Bhaskara forms (*viz.*, the predecessor of the first member of the cycle, as well as the last member), will have the same successor.

THEOREM. IV. *Two different cycles of Bhaskara forms are non-equivalent, and so all equivalent Bhaskara forms belong to the same cycle.*

* *Vide* p. 238, Cor. 3.

To prove this, we have to examine the structure of a complete cycle of these forms and perceive the relation between them and Gauss-forms.

Let $(k, l, k_1), (k_1, l_1, k_2), \ldots (k_s, l_s, k_{s+1}), \ldots \ldots$

be a cycle of Bhaskara forms with the l's all positive.

If $l_r < \sqrt{N}$, the form (k_r, l_r, k_{r+1}) is readily seen to be a Gauss-form satisfying the conditions $\sqrt{N} - l_r < | k_r | < \sqrt{N} + l_r$, where $N \equiv l_r^2 - k_r k_{r+1}$ (*vide* p. 74. Mathews' Theory of Numbers).

If $l_r > \sqrt{N}$, the proper unitary substitution $\begin{pmatrix} 1, & \pm 1^* \\ 0, & 1 \end{pmatrix}$ transforms (k_r, l_r, k_{r+1}) into the equivalent form $(k_r, l_r - | k_r |, k_r')$, where

$$| k_r' | = 2 l_r - | k_r | - | k_{r+1} | .$$

Since $\sqrt{N} - l_r + | k_r | < | k_r | < \sqrt{N} + l_r - | k_r |$,

the latter form is a reduced Gauss-form.

Thus, we see that with every Bhaskara form, there is always an associated equivalent Gauss-form,† the associated form being identical with the Bhaskara form when the middle co-efficient is less than $\sqrt{N}$.

Now let $(k_r, l_r, k_{r+1}), (k_s, l_s, k_{s+1})$ be two forms in Bhaskara's cycle (where l_r, l_s are each less than $\sqrt{N}$, so that the forms are also Gauss-forms) with the intervening forms having their middle co-efficients all greater than $\sqrt{N}$. We can now replace all these intervening forms by their equivalent Gauss-forms, and form the sequence :

$$(k_r, l_r, k_{r+1}), (k_{r+1}, l_{r+1} - | k_{r+1} |, k_{r+1}'), (k_{r+2}, l_{r+2} - | k_{r+2} |, k_{r+2}')$$
$$\ldots \ldots (k_{s-1}, l_{s-1} - | k_{s-1} |, k_{s-1}'), (k_s, l_s, k_{s+1})$$

in which all the forms are Gauss's.

We shall now interpolate between any two consecutive forms of the above sequence (except the first two) another Gauss-form and thus get a complete succession of unique ‡ Gauss-forms from (k_r, l_r, k_{r+1}) to (k_s, l_s, k_{s+1}). For example,

between $(k_{r+1}, l_{r+1} - | k_{r+1} |, k'_{r+1})$

and $(k_{r+2}, l_{r+2} - | k_{r+2} |, k'_{r+2})$

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* The upper sign is to be taken when k_r is negative and the lower sign when k_r is positive.

† Not *vice-versa*.

‡ *Vide* p. 76, Art. 70. Mathews' Theory of Numbers.

we may put in the Gauss-form

$$(k'_{r+1},\ l_{r+1} - |\,k_{r+2}\,|,\ k_{r+2})^{*}$$

The latter form cannot be also a Bhaskara form, for,

$$|\,k'_{r+1}\,| - \tfrac{1}{2}\,|\,k_{r+2}\,| = 2\,l_{r+1} - |\,k_{r+1}\,| - \tfrac{3}{2}\,|\,k_{r+2}\,| > l_{r+1} - |\,k_{r+2}\,|$$

owing to the condition $|\,k_{r+1}\,| + \frac{1}{2}|\,k_{r+2}\,| < l_{r+1}$.

The above process enables us, in general, to convert a Bhaskara cycle into a unique Gauss cycle and *vice-versa*.

Two different cycles of Bhaskara forms † cannot evidently be transformed into the same Gauss-cycle and are therefore non-equivalent. Thus, all equivalent Bhaskara forms belong to the same period.

THEOREM V. *If L^2 be the integral square nearest to N and* K $= L^2 - N$, *then* (K, L, 1) *where* L *is positive, is a Bhaskara form.*

For, let $L^2 < N$, then $N - L^2 < (L+1)^2 - N$

$$\begin{aligned}
&\textit{i.e.,} & N &< L^2 + L + \tfrac{1}{2}\\
&\textit{i.e.,} & N - L^2 - \tfrac{1}{2} &< L\\
&\textit{i.e.,} & -K - \tfrac{1}{2} &< L.\\
\therefore\ &\text{Squaring,} & K^2 + K + \tfrac{1}{4} &< L^2\\
&\textit{i.e.,} & K^2 + \tfrac{1}{4} &< L^2 - K\\
& & &< N
\end{aligned}$$

Hence $\frac{K^2}{4} + 1 < K^2 + \frac{1}{4} < N$, and the criteria for a Bhaskara form are satisfied.

The case $L^2 > N$ can also be similarly treated.

Now, putting $a = L$, $b = -1$, $a_1 = 1$, $b_1 = 0$ in the Lemma on p. *231*, we find that (K, L, 1) and (1, 0, $-$N) are equivalent forms. But the Bhaskara forms obtained with the help of a pair of known integral roots (a, b) of the equation

$$x^2 - Ny^2 = k$$

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* To prove this to be a Gauss-form, we have only to use

$$l_{r+1} > |\,k_{r+2}\,| + \tfrac{1}{2}\,|\,k_{r+1}\,|$$

and cor. 3 p. *238*.

† Two different Bhaskara cycles cannot have any common form ; for, if one form is common, all its predecessors and successors also must be common, so that the two cycles entirely coincide.

are all equivalent to $(1, 0, -N)$. These form a cycle, which includes as a member the particular form $(K, L, 1)$.

Thus, we ultimately get one pair of integral roots of the equation $x^2 - Ny^2 = 1$.

From this single pair, an infinite number can be obtained by the principle of composition mentioned at the beginning of this paper. This completes the proof of Bhaskara's cyclic method.

4. Relation of the Cyclic Method to the Irregular Continued Fraction.

Just in the same way as in Gauss's Theory (*vide* Mathews, pp. 68, 78, 79), each form of a Bhaskara period, say (k, l, k_1) is transformed into the next following (k_1, l_1, k_2) by the substitution

$$\begin{pmatrix} 0, & 1 \\ -1, & -\delta_1 \end{pmatrix} \text{ where } \delta_1 = \frac{l + l_1}{k_1}.$$

If ω_1, ω_1' be the principal roots (*i.e.* roots numerically less than 1) of these forms,

$$\omega_1 = \frac{1}{-\omega_1' - \delta_1} = \frac{-1}{\delta_1 + \omega_1'}.$$

Similarly, if $\begin{pmatrix} 0, & 1 \\ -1, & -\delta_2 \end{pmatrix}$ transforms (k_1, l_1, k_2) into the next form of the period,

$$\omega'_1 = \frac{-1}{\delta_2 + \omega'_2}$$

and so on.

Hence
$$\omega_1 = \frac{-1}{\delta_1 +}\,\frac{-1}{\delta_2 +}\,\frac{-1}{\delta_3 +} \cdots \cdots$$

which is a recurring irregular continued fraction since the δ's recur with the forms.

Now, we are in a position to connect Bhaskara's algorithm with an irregular continued fraction.

To start with, (a, b) are the roots of $Ny^2 + k = x^2$; from this, we derive successively (a_1, b_1) the roots of $Ny^2 + k_1 = x^2$,

(a_2, b_2) ,, $Ny^2 + k_2 = x^2$,

and so on.

The corresponding quadratic forms are (*vide* p. 232. supra)

$$(k, l, k_1),\ (k_1, l_1, k_2),\ \ldots\ \ldots$$

with
$$\delta_1 = \frac{l + l_1}{k_1},\ \delta_2 = \frac{l_1 + l_2}{k_2},\ \ldots\ \ldots$$

(δ's being integers).

The principal root of the form (k, l, k_1) is

$$\frac{-l+\sqrt{N}}{k}$$

and we get

$$\frac{-l+\sqrt{N}}{k} = \frac{-1}{\delta_1 +}\;\frac{-1}{\delta_2 +}\;\ldots\;\ldots$$

We may write

$$\sqrt{N} = \cfrac{a}{\cfrac{a}{\sqrt{N}} - b + b}$$

$$= \cfrac{a}{b + \cfrac{k}{bN + a\sqrt{N}}} = \cfrac{a}{b + \cfrac{1}{a_1 + a(\sqrt{N} - l)/k}}\,*$$

$$= \frac{a}{b +}\;\frac{1}{a_1 +}\;\frac{-a}{\delta_1 +}\;\frac{-1}{\delta_2 +}\;\ldots\;\ldots$$

$$= \frac{a}{b +}\;\frac{1/a}{a_1/a +}\;\frac{-1}{\delta_1 +}\;\frac{-1}{\delta_2 +}\;\ldots\;\ldots\dagger$$

The successive convergents to this continued fraction are

$$\frac{a}{b},\ \frac{a_1}{b_1},\ \frac{a_2}{b_2},\ \ldots\ \ldots\ \frac{a_r}{b_r},\ \frac{a_{r+1}}{b_{r+1}},\ \frac{a_{r+2}}{b_{r+2}},\ \ldots\ \ldots;$$

for

$$a_2 = a_1\delta_1 - a = a_1 . \frac{l + l_1}{k_1} - a,$$

$$b_2 = b_1\delta_1 - b = b_1 . \frac{l + l_1}{k_1} - b,$$

$$\ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots$$

$$a_{r+2} = \delta_{r+1}\, a_{r+1} - a_r = a_{r+1} . \frac{l_r + l_{r+1}}{k_{r+1}} - a_r,$$

$$b_{r+2} = \delta_{r+1}\, b_{r+1} - b_r = b_{r+1} . \frac{l_r + l_{r+1}}{k_{r+1}} - b_r,$$

$$\ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots$$

---

* Since $a_1 = \frac{al + Nb}{k}$ (*Vide* p. 233.)

† For example,

$$\sqrt{67} = \frac{8}{1 +}\;\frac{1/8}{(-41/8) +}$$

$$\left[\frac{-1}{2 +}\;\frac{(-1)}{(-2) +}\;\frac{(-1)}{(-9) +}\;\frac{(-1)}{(-2) +}\;\frac{(-1)}{2 +}\;\frac{(-1)}{(-5) +}\;\frac{(-1)}{16 +}\;\frac{(-1)}{(-5)}\right]$$

the part within the brackets being the recurring portion of the C. F.

Thus, Bhaskara's method of finding successively the roots (a_1, b_1), (a_2, b_2), ... is equivalent to finding the successive convergents of the irregular continued fraction

$$\sqrt{N} = \frac{a}{b+} \frac{1/a}{a_1/a+} \frac{-1}{\delta_1 +} \frac{-1}{\delta_2 +} \cdots \cdots$$

This shows clearly that Bhaskara's method is not related to the Lagrangean or Eulerian method involving the simple C. F. expansion of $\sqrt{N}$, as is generally believed to be the case by T. L. Heath and others.

One point, however, is worth remarking. Lagrange's chain of reductions by which he is able to prove the important theorem that any integral solution of $x^2 - Ny^2 = k$ ($|k| > \sqrt{N}$), must be deducible from the solution of one or other of a finite group of equations of the type $x^2 - Ny^2 = k'$ ($|k'| < \sqrt{N}$) is almost the same as Bhaskara's, except for a slight variation in the choice of roots of a linear indeterminate equation. Though the methods of reduction appear to be more or less the same, Bhaskara and Lagrange apply them as it were at opposite ends with opposite aims in view. While Bhaskara tries to solve $x^2 - Ny^2 = 1$ from a known solution of the equation $x^2 - Ny^2 = k$, Lagrange attempts the converse problem of solving $x^2 - Ny^2 = k$ with the known solutions of $x^2 - Ny^2 = k'$ where $|k'| < \sqrt{N}$, (*vide* Chrystal's Text-Book of Algebra, pp. 482—485).

5. Bhaskara and Fermat.

Since Bhaskara clearly mentions the existence of an infinite number of solutions (पदानामानन्त्यम्), it cannot be said that Fermat was the first* to assert that the equation $x^2 - Ny^2 = 1$ (N being a non-square integer) always has an unlimited number of solutions in integers. Further, while Fermat claims that he can prove his assertion 'by the method of *descente* applied in a quite special manner,' we have presumably in Bhaskara's work a similar method. For, the cyclic method possesses the characteristics of the method of descent, starting as it does from the equation $x^2 - Ny^2 = k$ with known roots and

* This remark is due to T L. Heath (*Vide* his *Diophantus*, p. 285.)

successively reducing $|k|$, if $|k| > \sqrt{N}$, to a value less than $\sqrt{N}$, and thereafter other k's all numerically less than $\sqrt{N}$, finally ending in a cycle containing the desired *Kshepa*, unity. This anticipation of Fermat is perhaps sufficient in itself to justify Hankel's praise of the Indian method that 'it is certainly the finest thing achieved in the Theory of Numbers before Lagrange.' Remarkably enough, one of Bhaskara's worked examples, *viz*., to find the roots of the equation $61y^2 + 1 = x^2$ is just one of those proposed by Fermat in his letter to Frénicle, of February, 1657 (Quel est, par example le plus petit quarré qui, multipliant 61, en prenant l' unité, fasse un quarré ?)* While Fermat chose this equation evidently for its difficulty, Bhaskara's reason for giving it was just the opposite, *viz*., to point out that it is one of those *favourable* cases where the solution can be made to depend upon the solution of $Ny^2 \pm 4 = x^2$. For, from the solution ($x = 8$, $y = 1$) of the equation $61y^2 + 3 = x^2$, Bhaskara's method immediately gives in the next stage of reduction the roots

$$x = 39, \quad y = 5$$

for the equation

$$61y^2 - 4 = x^2.$$

Applying repeatedly the principle of composition of forms, Bhaskara obtains

$$x = 1766319049, \quad y = 226153980$$

as the required roots.

Finally, it may be remarked that the early recognition by the ancient Hindus† of the importance of the fundamental equations $Ny^2 \pm 1 = x^2$ and $Ny^2 \pm 4 = x^2$ in the solution of the indeterminate equations of the second degree testifies to their instinctive grasp of the true nature of the problem.

MADRAS,
20*th June* 1930.

A. A. KRISHNASWAMI AYYANGAR, M.A., L.T.

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* *Oeuvres de Fermat* II, p. 334

† In the same way, in the case of the linear indeterminate equation, they were the first to recognise the fundamental nature of the equation $ay \pm 1 = bx$.

Solutions.

Question 1441.

(K. J. SANJANA, M.A.) :—A chord of length λ of a hyperbola lies along the straight line L and l is any point on the line; f, f' are the foci of the curve and Σ, Σ' its asymptotes. If from the point l tangents T and T$'$ are drawn to the hyperbola, prove that

$$\sin(\mathrm{LT})\sin(\mathrm{LT}')\sin^2(\Sigma\Sigma')\,\overline{lf}.\overline{lf'} = \sin^2(\mathrm{L}\Sigma)\sin^2(\mathrm{L}\Sigma')\,\lambda^2.$$

The property holds evidently for the ellipse and its imaginary asymptotes; examine what it indicates in the case of the parabola.

[*Notation.* $\overline{xy}$ denotes the distance between the points x and y, and (XY) the angle between the lines X and Y.]

Solution by T. R. Raghava Sastri.

The following theorems are used; the reader is referred to Casey's *Analytical Conics*, p. 225 for their proofs.

(1) In a central conic $\cos(\mathrm{TT}') = \dfrac{\rho^2+\rho'^2-4a^2}{2\rho\rho'}$ where ρ and ρ' stand for $\overline{lf}$ and $\overline{lf'}$.

(2) In a central conic, if the chord of length λ envelopes a confocal, then $\lambda^2 = \dfrac{4(a'^2-a^2)}{a^2b^2}\mathrm{CD}^4$ where a' is the semi-major axis of the confocal and CD the semi-diameter of the original conic parallel to the chord.

Now $\sin(\mathrm{LT})\sin(\mathrm{LT}') = \frac{1}{2}[\cos(\mathrm{LL}')-\cos(\mathrm{TT}')]$ where L$'$ is the other tangent from l to the confocal touching L.

But from (1) $$\cos(\mathrm{TT}') = \frac{\rho^2+\rho'^2-4a^2}{2\rho\rho'}$$

and similarly $$\cos(\mathrm{LL}') = \frac{\rho^2+\rho'^2-4a'^2}{2\rho\rho'}.$$

Hence

$$\sin(\mathrm{LT})\sin(\mathrm{LT}') = \tfrac{1}{2}[\cos(\mathrm{LL}')\sim\cos(\mathrm{TT}')] = \frac{a^2\sim a'^2}{\rho\rho'}$$

and from (2)

$$\frac{a^2 \sim a'^2}{\rho\rho'} = \frac{a^2b^2\lambda^2}{4\rho\rho'\,CD^4}.$$

Hence

$$\sin(LT)\sin(LT') = \frac{a^2b^2\lambda^2}{4\rho\rho'\,CD^4}$$

or

$$\frac{\overline{lf}\,.\,\overline{lf'}\,\sin(LT)\sin(LT')}{\lambda^2} = \frac{a^2b^2}{4\,CD^4},$$

a constant so long as the straight line L is fixed in direction. Now taking l at the centre of the hyperbola and drawing L in the fixed direction we get the given theorem.

In the case of a parabola, if L and M are parallel chords of lengths λ and λ' we have from the general theorm

$$\frac{\overline{lf}\,.\,\overline{lf'}\,\sin(LT)\sin(LT')}{\lambda^2} \bigg/ \frac{\overline{mf}\,.\,\overline{mf'}\,\sin(MT)\sin(MT')}{\lambda'^2} = 1$$

as the expression $\dfrac{\sin^2(L\Sigma)\sin^2(L\Sigma')}{\sin^2(\Sigma\Sigma')}$

is the same for both. Remembering that f' is at infinity and therefore that $\dfrac{\overline{lf'}}{\overline{mf'}} = 1$ we get from the above

$$\frac{\overline{lf}\,\sin(LT)\sin(LT')}{\lambda^2}$$

is a constant so long as L is fixed in direction (as in fact, we can get directly); we can now take l to be any convenient point, say, at the vertex, in which case the theorem becomes

$$\frac{\overline{lf}\,\sin(LT)\sin(LT')}{\lambda^2} = \frac{\sin^4(LA)}{4p}$$

where A is the axis and p the parameter of the parabola.

Question 1451.

(N. DURAIRAJAN) :—If G be the centre and S a focus of the maximum inscribed ellipse of a triangle and if GS be produced to S′ so that $GS' = \sqrt{3}\,GS$, then prove that

$$S'A\,.\,S'B\,.\,S'C = GA\,.\,GB\,.\,GC$$

Solution by T. R. Raghava Sastri.

Let the given triangle ABC and its maximum inscribed ellipse be the orthogonal projection of a triangle A′B′C′ and its incircle. Let the axis of projection be GL and θ the angle of projection. Also let the angles LGA′ and LS′A′ be α and ϕ respectively.

$$GS'^2 = 3\,GS^2 = 3\,(a^2 - b^2)$$

$$= 3\,(a^2 - a^2\cos^2\theta) = 3a^2\sin^2\theta.$$

Also $A'G = 2a$ and

$$\frac{GS'}{A'G} = \frac{\sin(\phi - \alpha)}{\sin\phi}$$

$$= \cos\alpha - \cot\phi\,\sin\alpha$$

$$= \frac{\sqrt{3}}{2}\sin\theta = \sin\frac{\pi}{3}\sin\theta \qquad \ldots \quad (1)$$

Now $S'A^2 = S'A'^2 - AA'^2 = 4a^2\,(\sin^2\alpha\,\operatorname{cosec}^2\phi - \sin^2\alpha\,\sin^2\theta) \qquad \ldots \quad (2)$

since $\dfrac{S'A'}{A'G} = \dfrac{\sin\alpha}{\sin\phi}$ and $AA' = LA\sin\theta$.

From (2) $S'A^2 = 4a^2\,[\sin^2\alpha\,(1 + \cot^2\phi) - \sin^2\alpha\,\sin^2\theta)$

$$= 4a^2\,[\sin^2\alpha + (\cos\alpha - \sin\frac{\pi}{3}\sin\theta)^2 - \sin^2\alpha\,\sin^2\theta\,]$$

from (1)

$$= 4a^2\left[1 - 2\cos\alpha\,\sin\frac{\pi}{3}\sin\theta + \sin^2\theta\left(\sin^2\frac{\pi}{3} - \sin^2\alpha\right)\right]$$

$$= 4a^2\left[1 - \sin\left(\frac{\pi}{3} - \alpha\right)\sin\theta\right]\left[1 - \sin\left(\frac{\pi}{3} + \alpha\right)\sin\theta\right]$$

To get the values of $S'B^2$ and $S'C^2$ we must put in the above for α, $\alpha + 120^\circ$ and $\alpha + 240^\circ$ respectively.

Hence $S'A^2 . S'B^2 . S'C^2 = 64a^6\,[1 - \sin^2\alpha\,\sin^2\theta\,]$

$$\left[1 - \sin^2\left(\alpha + \frac{2\pi}{3}\right)\sin^2\theta\right]\left[1 - \sin^2\left(\alpha + \frac{4\pi}{3}\right)\sin^2\theta\right]$$

$$= GA^2 . GB^2 . GC^2$$

since $GA^2 = GA'^2 - LA'^2\sin^2\theta = 4a^2\,(1 - \sin^2\alpha\,\sin^2\theta)$.

Question 1473.

(A. A. KRISHNASWAMI AIYANGAR):—If ABC, A′B′C′ be two triangles inscribed in a conic and XYZ be the triangle formed by AA′, BB′ and CC′ (BB′, CC′ meeting in X and so on); prove that, if any triangle PQR inscribed in the same conic be in perspective with two of the three triangles ABC, A′B′C′, XYZ, it is also in perspective with the third triangle.

Solution by B. Ramamurthi.

Let AA′, BB′, CC′ meet QR, RP, PQ in T_1 T_2 T_3 respectively. Project the conic into a circle and T_1 T_2 into the line at infinity. Now AA′ is parallel to QR, BB′ to RP. Hence XYZ will be in perspective with PQR if CC′ is parallel to PQ. We have to prove now that

(1) CC′ is parallel to PQ if PA, QB, RC and PA′, QB′, RC′ are concurrent.

(2) If CC′ is parallel to PQ and PA, QB, RC, are concurrent, then PA′, QB′, RC′ are concurrent.

AA′ is parallel to QR, BB′ to RP and the circumscribing conic is a circle.

Hence PA′, QB′ are respectively the isogonal conjugates of PA, QB. So with the data of (1) it follows that PC, PC′ are isogonal conjugates. Hence CC′ is parallel to PQ. With the data of (2) PA, QB, RC have for their isogonal conjugates PA′, QB′, RC′ and PA, QB, RC are concurrent. Hence PA′, QB′, RC′ are also concurrent.

Question 1479.

(T. TOTADRI IYENGAR):—Shew that

$$\prod_{m=0}^{n-1} (1 + 2\cos 3^m\theta) = 1 + 2\sum_{m=1}^{(3^n-1)/2} \cos m\theta$$

and deduce the well-known result that any integer from 1, 2, 3, ... up to $(3^n - 1)/2$ can be uniquely made up of 1, 3, 3^2, ... 3^{n-1} the parts being taken positively or negatively but without repetition.

Solution by B. B. Bagi.

Multiply the product

$$F(x) = (1 + x + x^2)(1 + x^3 + x^6)(1 + x^9 + x^{18}) \ldots \ldots$$

to n factors by $(1 - x)$; it then reduces to $1 - x^{3^n}$.

$$\therefore\ F(x) = (1 - x^{3^n}) \div (1 - x)$$
$$= 1 + x + x^2 + x^3 + \ldots + x^{3^n-1}.$$

Dividing this identity by $x^{(3^n-1)/2}$ we change it into

$$(1+x+x^{-1})(1+x^3+x^{-3})(1+x^9+x^{-9}) \dots \dots \text{ to } n \text{ factors}$$
$$= 1+(x+x^{-1})+(x^2+x^{-2})+ \dots \dots (x^{(3^n-1)/2}+x^{-(3^n-1)/2}).$$

By putting $x = \cos\theta + i\sin\theta$ in this identity the first part is obtained and by interpreting the identity we get the second part.

Also solved by G. V. Krishnaswami Aiyangar and T. Totadri Aiyangar, and partially by V. A. Mahalingam.

Question 1488.

(S. L. MALURKAR) :—Shew that

$$\cos n\theta = \frac{2^n\, n!}{(2n)!} \sin\theta \left(\frac{1}{\sin\theta}\cdot\frac{d}{d\theta}\right)^n \left(\sin\theta\right)^{2n-1},$$

where n is a positive integer.

Solution by (1) *T. Totadri and* (2) *S. D. Chowla,*

(1) If n is a positive integer Rodrigue's Theorem states (*Vide* Edwards: *Differential Calculus*, p. 77, Ex. 38), that

$$\sin n\theta = \frac{n}{1.3.5 \dots (2n-1)}\cdot\left(\frac{1}{\sin\theta}\cdot\frac{d}{d\theta}\right)^{n-1}\left(\sin\theta\right)^{2n-1}.$$

Operating on both sides by $\frac{1}{\sin\theta}\cdot\frac{d}{d\theta}$, multiplying by $\sin\theta$ and noting that

$$\frac{1}{1.3.5 \dots (2n-1)} = \frac{2^n n!}{(2n)!},$$

the result is got.

(2) If we write μ for $\cos\theta$, this is the same as Jacobi's well-known transformation

$$\frac{d^{n-1}}{d\mu^{n-1}} \sin^{2n-1}\theta = (-1)^{n-1}\,\frac{1.3 \dots (2n-1)}{n}\sin n\theta.$$

An extremely simple proof is given in the *Proc. Lond. Math. Soc*, 1924 (near the end of the Records of Meetings).

Also solved by R. Kothandaraman and K. V. Krishna Sastri.

Question 1545.

(K. J. SANJANA, M.A., AND G. R. JANI, B.A.):—Resolve the following numbers into factors :—

$$N_1 = 3^{52} + 2^{26},\ N_2 = 3^{27} + 2^{54} \text{ and } N_3 = 5^{25} - 2^{50}.$$

Solution by Kaikobad F. Vakil

The numbers proposed are particular cases of those dealt with by Prof. Sanjana in his paper "On Factors of Numbers of the form $\frac{x^{(2n+1)k} \pm 1}{x^k \pm 1}$" where kx is a perfect square, published in the *Proceedings of the Edinburgh Math. Society*, Vol. XXVI (1907-1908). The cases when k is 2, 3, 5 and 7 are fully treated; for odd prime values of k the upper or lower sign is to be taken according as k is of form $4p - 1$ or $4p + 1$.

(1) Writing N_1 in the form $9^{26} + 2^{26}$ or $2^{26}\{(9/2)^{26} + 1\}$, we see that $26 = 2(2.6 + 1)$ and that $kx = 9$, a perfect square.

Hence the number will have four factors at least and these may be made integral by multiplying out by 2^{26}. Following the process given in the paper refered to above, we have

$$\frac{x^{26}+1}{x^2+1} \equiv x^{24} - x^{22} + x^{20} \dots \dots + x^4 - x^2 + 1$$
$$= (x^{12} + x^{11} - x^{10} - x^9 + x^8 + x^7 - x^6 + x^5 + x^4 - x^3 - x^2 + x + 1)^2$$
$$- 2x(x^{11} - x^9 + x^7 + x^4 - x^2 + 1)^2.$$

Changing x to x/y and multiplying out by y^{26}, we get the identity :

$$(x^{26} + y^{26})/(x^2 + y^2)$$
$$= (x^{12} + x^{11}y - x^{10}y^2 - x^9y^3 + x^8y^4 + x^7y^5 - x^6y^6 + x^5y^7 + x^4y^8 - x^3y^9$$
$$- x^2y^{10} + xy^{11} + y^{12})^2 - 2xy(x^{11} - x^9y^2 + x^7y^4 + x^4y^7 - y^2y^9 + y^{11})^2$$

$= A^2 - 2xy\ B^2$, calling the brackets A and B respectively. This identity can be verified algebraically.

Now here

$$x = 9,\ y = 2 \text{ and } 2xy = 36 = 6^2;\ \text{also, } x^2 + y^2 = (x + y)^2 - 2xy$$
$$= (9 + 2)^2 - 6^2 = (11 - 6)(11 + 6).$$

Therefore

$$N_1 = 5.17\,(A - 6B)(A + 6B).$$

On actual calculation we get

$$A \equiv 9^{12} + 9^{11}.2 - 9^{10}.2^2 - 9^9.2^3 + 9^8.2^4 + 9^7.2^5 - 9^6.2^6 + 9^5.2^7$$
$$+ 9^4.2^8 - 9^3.2^9 - 9^2.2^{10} + 9.2^{11} + 2^{12}$$

$= 328,961,748,727;$

$B = 9^{11} - 9^9 . 2^2 + 9^7 \; 2^4 + 9^4 . 2^7 - 9^2 . 2^9 + 2^{11} = 29, 908, 705, 541,$
so that

$6\,B = 179, 452, 233, 216.$ Thus finally we have

$3^{52} + 2^{26} = 5 . 17 . (149, 509, 515, 481)\;(518, 413, 981, 973).$

(2) Writing N_2 in the form $4^{27} + 3^{27}$, we see that $27 = 3\,(2\;4+1)$ and that

$$kxy = 3 . 4\;\; 3 = 6^2.$$

This number will have five factors at least; for

$$x^{27} + y^{27} = (x^9 + y^9)\,(x^{18} - x^9y^9 + y^{18}) = (x^9 + y^9)\,\{ (x^9 + y^9)^2 - 3x^9y^9 \}$$
$$= (x^9 + y^9)\,\{ (x^9 + y^9)^2 - 3xy \,.\, x^8y^8 \}$$
$$= (x^9 + y^9)\,(x^9 + y^9 - 6x^4y^4)\,(x^9 + y^9 + 6x^4y^4),$$

and similarly

$$x^9 + y^9 = (x^3 + y^3)\,\{ (x^3 + y^3)^2 - 3xy \,.\, x^2y^2 \}$$
$$= (x^3 + y^3)\,(x^3 + y^3 - 6xy)\,(x^3 + y^3 + 6xy).$$

We can also resolve

$x^3 + y^3$ into $(x + y)\,(x + y - 6)\,(x + y + 6) = 7 . 1 . 13$

in the present case. Thus we get $3^{27} + 2^{54}$

$$= 7 . 13 , (4^3 + 3^3 - 72)\,(4^3 + 3^3 + 72)$$
$$\times (4^9 + 3^9 - 6 . 4^4 . 3^4)\,(4^9 + 3^9 + 6 . 4^4 . 3^4)$$
$$= 7 . 13 . 19 . 163 . (157, 411)\,(406, 243)$$
$$= 7 . 13 . 19 . 109 . 163 . 3727 . 157411,$$

and the resolution is ultimate.

(3) Writing N_3 in the form $5^{25} - 4^{25}$, we see that $25 = 5\,(2 . 2+1)$ and that

$$kxy = 5 . 5 . 4 = 10^2.$$

This number will have four factors at least; for

$$x^{25} - y^{25} = (x^5 - y^5)\,(x^{20} + x^{15}y^5 + x^{10}y^{10} + x^5y^{15} + y^{20})$$
$$= (x^5 - y^5)\,\{ (x^{10} + 3x^5y^5 + y^{10})^2 - 5x^5y^5\,(x^{10} + 2x^5y^5 + y^{10}) \}$$
$$= (x^5 - y^5)\,\{ (x^{10} + 3x^5y^5 + y^{10})^2 - 10^2x^4y^4\,(x^5 + y^5)^2 \}$$
$$= (x^5 - y^5)\,\{ x^{10} + 3x^5y^5 + y^{10} - 10x^2y^2\,(x^5 + y^5) \}$$
$$\times \{ x^{10} + 3x^5y^5 + y^{10} + 10x^2y^2\,(x^5 + y^5) \},$$

and similarly

$$x^5 - y^5 = (x - y)\,\{ x^2 + 3xy + y^2 - 10\,(x + y) \}$$
$$\{ x^2 + 3xy + y^2 + 10\,(x + y) \}.$$

As $x - y = 1$, we get $5^{25} - 2^{50}$

$$= \{ (5^2 + 3 . 5 . 4 + 4^2 - 10\,(9) \}\;\{ 5^2 + 3 . 5 . 4 + 4^2 + 10\,(9) \}$$
$$\{ 5^{10} + 3 . 5^5 . 4^5 + 4^{10} - 10 . 5^2 . 4^2\,(5^5 + 4^5) \}$$
$$\times \{ 5^{10} + 3 . 5^5 . 4^5 + 4^{10} + 10 . 5^2 . 4^2\,(5^5 + 4^5) \}.$$
$$= (101 - 90)\,(101 + 90)\,(20{,}414{,}201 - 16{,}596{,}000)$$
$$(20{,}414{,}201 + 16{,}596{,}000)$$
$$= 11 . 191 . (3{,}818{,}201)\,(37{,}010{,}201);$$

This resolution is also ultimate.

Questions for Solution.

Proposers of Questions are requested to send their own solutions along with their questions.

1569 (R. VAIDYANATHASWAMY) :—(1) Meaning by a *pedal tetrad* on a central conic, a set of four points the normals at which are concurrent, shew that if $abcd$, $ab'c'd'$ are pedal tetrads, then the triads bcd, $b'c'd'$ are apolar.

(2) If the normals at b, c to a central conic pass through the centre of curvature at a, show that bc is the first polar of a in regard to the three other vertices of the inscribed rectangle which has one vertex at a.

(3) Shew that the four other points the centres of curvature at which lie on the normal at a form a self-apolar pedal tetrad.

1570. (A. A. KRISHNASWAMI AIYANGAR, M.A., L.T.):—Show that a convex polygon of n sides can be divided into $n-2$ non-overlapping triangles (by drawing $n-2$ diagonals) in

$$\frac{(2n-4)!}{(n-1)!\,(n-2)!}$$

ways. Trace the connection between this result and the number of lines in a space of n-dimensions which cut $2(n-1)$ linear spaces of $(n-2)$ dimensions.

1571. (V. RAMASWAMI AIYAR, M.A.):—Let M be the foot of the perpendicular drawn from focus S on the tangent at a fixed point P of a conic. If PP′, SS′ be perpendiculars drawn on a variable tangent of the conic, show that PP′.SS′ varies as MS'^2.

1572. (S. AUDINARAYANAN) :—If P, Q are two points in the plane of a triangle ABC such that

$$\frac{3}{PQ}=\frac{1}{PA'}+\frac{1}{PB'}+\frac{1}{PC'}=\frac{1}{QA'}+\frac{1}{QB'}+\frac{1}{QC'}$$

where A′B′C′ are the points where PQ meets the sides of ABC, then for every point P there are two points Q, Q′, and the triad PQQ′ is a closed triad and is in sextaple perspective with ABC.

15326

LIST OF JOURNALS & BOOKS RECEIVED IN THE LIBRARY

during the months of June and July 1930.

1 Acta Mathematica, **54**, 1, 2.
2 American Journal of Mathematics, **52**, 2.
3 American Mathematical Monthly, **37**, 5.
4 Annales de l'Ecole Normale Supereure, **66**, 1, 2, 3, 4, 5, 6.
5 Annals of Mathematics, **31**, 2.
6 Astrophysical Journal, **71**, 2.
7 Bulletin de la classe des Sciences Physiques et Mathematiques, **4**, 4.
8 Bulletin des Sciences Mathematiques, **54**, April and May, 1930,
9 Bulletin of the American Mathematical Society, **36**, 5.
10 Crelle's Journal, **162**, 1, 2.
11 Jahrbuch über die Fortschritte der Mathematik, **39**, 1 to 8.
12 Mathematical Gazette, **15**, 206, 207 (2 copies each).
13 Mathematische Annalen, **103**, 2.
14 Messenger of Mathematics, **58**, 11, 12.
15 Mysore University Journal, **4**, 1.
16 Monthly Notices of the Royal Astronomical Society, **90**, 5, 6, 7.
17 Philosophical Magazine, **9**, 59, 60.
18 Philosophical Transactions of the Royal Society, **229**, pp. 87—124, 125 to 162, 183 to 204, 205 to 253.
19 Popular Astronomy, **38**, 2, 3, 4, 5, 6.
20 Proceedings of the Cambridge Philosophical Society, **26**, 2.
21 Proceedings of the London Mathematical Society, **31**, 1, (3 copies).
22 Proceedings of the Physico-Mathematical Society of Japan, **12**, 4, 5.
23 Proceedings of the Royal Society of London, **127**, 805, 806. (2 copies each).
24 Quarterly Journal of Mathematics, **1**, 1.
25 Revue Semestrielle des Publications Mathematiques, Vol. 34.
26 Wiskundige opgaven met de Oplossungen, **4**.

Books and Pamphlets.

1 On the Geometry of the Weddle Surface, by Walter Lee Moore.
2 On the Summability of Fourier, Bessel, and Dini Expansions, by Hemphill Moffett, Hosford. (2 copies).
3 The Summability of Birkhoff Series, by Nelson Bush Coffkwright.
4 Recent Theories of Sovereignty, by Su Ching Chen
5 Vigor in soybeans as affected by hybridity and in inhibition of pubescence, by Collins Veatch.

Publication de La Faculte des Sciences.

1 Reflexion des ondes Acoustiques sur un Plan, No. 117.
2 Mereni Doby Kyvu Gravitacnich Vah, No. 118.
3 Buzeni Kratkych Elektroma gnetickych ven Dvoumrizkovymi Lampami, No. 120.
4 Proprietes Projectives du Contact, No. 121.

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